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Kai Wang

Model-Based Predictive Control Under Uncertainty

Doctoral thesis

NTNU
Norwegian University of Science and Technology
Thesis for the Degree of Philosophiae Doctor
Faculty of Information Technology and Electrical Engineering
Department of Engineering Cybernetics

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Summary

This thesis focuses on model-based predictive control under uncertainty, with a specific emphasis on systems affected by uncertainty characterized by worst-case geometric bounds and probability distributions. The research topic is motivated by the necessity for computationally efficient controllers tailored for uncertain dynamical systems, where safety of the controlled systems needs to be guaranteed to an acceptable level. A collection of novel algorithms addressing robust and stochastic model predictive control (MPC) problems has been reported.

In contrast to nominal MPC, which is widely employed in many control scenarios, robust and stochastic MPC have encountered fewer real-world applications primarily due to their numerical and theoretical challenges. However, uncertainty is a critical factor in many control scenarios. In MPC community, uncertainty is primarily addressed through two approaches: robust MPC and stochastic MPC. Robust MPC guarantees that all possible future state and control trajectories adhere to constraints while minimizing the "worst-case" cost or some generalized costs. In contrast, stochastic MPC aims to minimize an expectation cost or some generalized costs while incorporating chance (probabilistic) constraints. This approach allows the controlled system to deviate from constraints to an acceptable extent. The primary focus of this thesis is dedicated to the design of these two classes of approaches. Although strictly speaking, robust and stochastic MPC belong to distinct categories, robust MPC tends to yield conservative control policies since it disregards the stochastic information about the bounded uncertainty under consideration, if such stochastic information is available. Additionally, due to the utilization of chance constraints, stochastic MPC is capable of handling stochastic uncertainty with unbounded

support. On the other hand, addressing the expectation cost and chance constraints, along with analyzing the control-theoretic properties of the controlled systems, present greater challenges in stochastic MPC compared to robust MPC. Similar to nominal MPC, the key properties to examine when analyzing these MPC approaches include the stability, optimality, and constraint satisfaction of the closed-loop controlled systems.

For scenarios where uncertainty is bounded and no further stochastic information is available, this thesis introduces a robust MPC controller using tubes, which is able to exponentially stabilize the controlled system. For scenarios where uncertainty is characterized by stochastic descriptions. Within this context, this thesis introduces two stochastic MPC controllers designed to stabilize the system while adhering to stage-wise (pointwise-in-time) chance constraints, commonly referred to as joint chance constraints in the stochastic MPC community. Subsequently, we investigate mission-wide (dynamic-joint) chance constraints over the state trajectory, which are, in generally, more meaningful in defining safety for engineering systems. In this regard, we offer a characterization of the exact solution to the mission-wide chance-constrained optimal control problems, and subsequently, we approximate these exact solutions using a stochastic MPC approach with shrinking time horizons.

Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of Philosophiae Doctor (PhD) at the Norwegian University of Science and Technology (NTNU). I carried out the presented work at the Department of Engineering Cybernetics (ITK) under the main supervision of Professor Sébastien Gros and co-supervision of Professor Mary Ann Lundteigen. My work has been supported by the Norwegian Research Council (NFR) through the project "Risk-Based Model Predictive Control" with grant number UV988976100.

Acknowledgements

First of all, I thank my supervisor professor Sébastien Gros for the fruitful discussions and for sparking the ideas that underpin much of the content within this thesis. I also thank Sébastien Gros for his patience during challenging phases of my research journey. I am deeply grateful for his backing of my research endeavors within the Department of Engineering Cybernetics (ITK) and for creating an exceptionally enjoyable research team. Moreover, I also owe thanks to my co-supervisor professor Mary Ann Lundteigen for her constructive feedback during my midterm evaluation.

I also want to thank professor Saša V. Raković from Beijing Institute of Technology for his collaboration on the paper and for his valuable professional advice on robust model predictive control using tubes. In addition, I owe thanks to Dr. Sixing Zhang, a postdoctoral researcher working with Saša V. Raković, for many fruitful discussions during the paper collaboration. I would also like to thank the members of our research group—Wenqi Cai, Shambhuraj Sawant, Dr. Dirk Reinhardt, and Dr. Akhil S. Anand—for their invaluable assistance both in my scientific endeavors and daily life at ITK.

Moreover, I would like to express my gratitude to my fellow PhD colleagues, Dr. Jiayu Zhou, Oliver Hasler, Kiet Tuan Hoang, Simon Hoff, who brought a lot of joy to my life at Trondheim. Oliver and Kiet have also collaborated with me on writing papers that are included in this thesis. I have shared many joyful moments with Jiayu and Wenqi, whether it is cooking together, playing games, or exploring Trondheim. Simon, my first colleague in the office, provided invaluable guidance and assistance as I acclimated to the new environment abroad. I owe immense gratitude to all the administrative staff at ITK, particularly Tove Johnsen and Berit Dahl, for their help with all administrative matters. In addition, I am deeply appreciative of a wide circle of friends I have made during my stay in Norway. I am not listing their names to avoid a lengthy enumeration, but we have enjoyed various activities like cross-country and downhill skiing, ice skating, cabin trips, parties, and more.

Finally, I want to thank my parents and sister for their unwavering support, encouragement, and love, which have been the cornerstone of my life's journey. Their faith in me has given me the strength and motivation to overcome challenges and pursue my academic aspirations. Their presence has made every achievement more meaningful, and I am eternally grateful to them for their sacrifices, understanding, and endless love.

Kai Wang
Trondheim, Norway
October 16, 2024

Contents

Chapter 1

Introduction

The presented thesis consists of a series of contributions on the topic of model-based predictive control in the presence of uncertainty. This chapter provides the background and motivation of this thesis. This chapter also covers the research topics and objectives of this thesis. At the end of this chapter, an overview of the thesis is presented based on the contributions which are outlined chapter-by-chapter.

1.1 Background and Motivation

Model predictive control (MPC) stands out as one of the most investigated methods in control theory for optimal control. At each time step, the MPC controller solves an optimal control problem. The optimal control problem utilises predictions of the future behaviour of a dynamic system on a finite time horizon, and it makes use of the current state estimate as its initial condition. Once the optimal predicted control sequence is chosen, the first control input of the optimal predicted control sequence is applied on the actual system until the next state estimate is obtained. The procedure is then repeated. As depicted in Figure 1.1, the fundamental principle behind the MPC controller entails utilizing a mathematical model of the system for prediction. Subsequently, an optimization problem is solved using this model to determine the optimal control input.

Conventional control techniques such as proportional-integral-derivative (PID) control typically suffice for single-input single-output systems with low-order dynamics and without constraints taken into account [1]. In contrast, MPC stands out as an important model-based control strategy designed for complex multiple-input multiple-output control problems with constraints on the inputs and/or outputs.

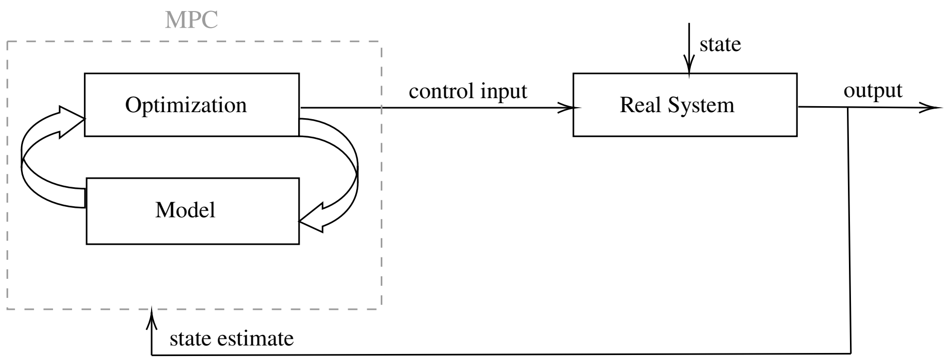

Figure 1.1: Streamlined block diagram illustrating an MPC-based control loop.

To summarize, the MPC approach offers several advantages over traditional control strategies where the following are the most important. There are other advantages too, but we outline the most important ones:

1. **Multivariable Control:** MPC can handle systems with multiple inputs and outputs. This capability is crucial in complex systems where interactions between variables need to be considered explicitly.
2. **Constraint Handling:** MPC systematically deals with input and state/output constraints such that safety of the controlled system is guaranteed.
3. **Inherent Robustness:** As a feedback controller, MPC exhibits a certain degree of inherent robustness to reject small disturbances, and it ensures stability of the closed-loop system.
4. **Integration with Advanced Control Strategies:** MPC can be integrated with other advanced control strategies, such as advanced uncertainty quantification and rejection techniques, as well as machine leaning and adaptive control algorithms, to further enhance system performance and robustness.

At present, there is a well-established mathematical understanding of the MPC controller, enabling researchers and practitioners to systematically handle challenges such as feasibility, stability, and performance. For those with a particular interest, refer to the monographs and survey papers such as [2]–[10]. Due to these advantages, the MPC controller has been extensively applied in various industrial processes; for example, see [11], [6, Part III], and [7, Chapters 7–9]. Obviously, the success of MPC, or any model-based control approach, hinges on the quality of the employed prediction model, which should ideally reflect the real system as accurately as possible. Predictions derived from an inaccurate model can deteriorate control performance rather than improve it. As a result, the biggest disadvantage of MPC is the significant effort required to obtain a process model from system identification techniques.

MPC controllers are implemented based on the certainty-equivalent principle. The process model is typically given in a disturbance-free, deterministic form, so that the future predictions of the system is optimized as though neither external disturbances nor model mismatch were present. As a feedback controller, MPC is inherently able to reject small magnitudes of uncertainty despite not considering uncertainty explicitly [12]–[16]. This inherent robustness renders conventional MPC suitable for applications where safety is of lesser concern.

In safety-critical applications where uncertainty affects the system and the inherent robustness of conventional MPC is insufficient to mitigate this uncertainty, it becomes necessary to consider the influence of uncertainty in the MPC formulation. The block diagram of MPC under uncertainty is briefly sketched in Figure 1.2. In this regard, various safety-conscious MPC variants have been proposed to explicitly handle uncertainty. These include robust MPC [2, Chapter 3] [4, Part II] [17]–[20], stochastic MPC [4, Part III] [21]–[23], adaptive MPC [24]–[27], as well as more recent developments such as distributionally robust MPC [28]–[30] and learning-based MPC [31]–[35], all aimed at ensuring the system's safety to an acceptable level.

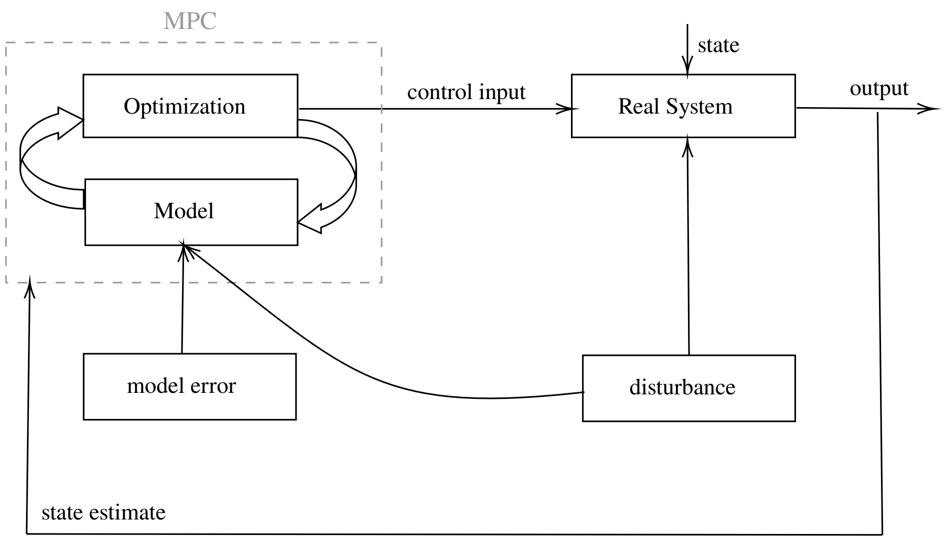

Figure 1.2: Streamlined block diagram of an MPC-based control loop that takes model mismatch and process disturbance into account.

These syntheses of MPC under uncertainty cope with various types of uncertainty that impact the system under consideration. Robust MPC synthesis considers the set-membership description of bounded uncertainty, in which only worst-case geometric bounds of the uncertainty are available. Stochastic MPC synthesis considers the uncertainty description taking the form of a specific probability distribution. Distributionally robust MPC synthesis addresses the uncertainty whose probability distribution belongs to a set of probability distributions. The synthesis of adaptive MPC aligns with the dual control paradigm, where the control inputs applied to an uncertain system serve the dual purpose of exploring and actively learning about uncertainty, while also directly controlling the system dynamics. Learning-based MPC is an ongoing research area that merges traditional MPC with machine learning techniques, leveraging gathered data. It involves using machine learning algorithms to train the MPC parameters in order to optimize control performance.

In this thesis, our primary focus is on robust and stochastic MPC syntheses. Robust (stochastic) MPC computes a control policy by repeatedly solving a finite–horizon robust (stochastic) optimal control problem in a receding horizon fashion. The robust optimal control problem formulation guarantees that all possible future state and control trajectories adhere to constraints while minimizing the "worst-case" cost or a generalized cost. In contrast, the formulation of the stochastic optimal control problem guarantees constraints in a probabilistic way while typically minimizing the expectation cost. In robust and stochastic optimal control problems, the decision variable is a control policy that consists of a sequence of control laws rather than a sequence of individual controls. The computational complexity associated with solving these problems is quite challenging. These numerical challenges encompass the necessity to model the uncertainty, as well as the complexity of selecting the optimal control policy from an infinite-dimensional function space. To implement robust and stochastic MPC controllers in real-world applications, it is necessary to utilize approximation methods to achieve a better balance between computational complexity and control performance in a theoretically rigorous manner. Consequently, robust and stochastic MPC remains an active area of research to date. In contrast to robust MPC, where significant progress has been achieved and the theoretical foundations are well-established, stochastic MPC research has not yet reached a full maturity.

1.2 Research Objectives

One of the state-of-the-art paradigms for robust MPC synthesis is tube MPC [18]. Tube MPC belongs to a class of set-theoretic control methods, in which the closed-loop tube model predictive controlled uncertain dynamics are obtained through the prediction of tubes rather than isolated trajectories. The deployed tubes capture the totality of possible realizations of future state sequences under a given control policy. Based on the key three tube MPC methods proposed for linear systems [36]–[38], many subsequent studies have emerged over the past two decades, as reviewed in Chapter 3. Each of these three methods has their own set of advantages and disadvantages. This motivates us to revisit these methods and propose an improved approach that leverages their strengths while mitigating their weaknesses, as shown in Chapter 3.

For stochastic MPC, solving the underlying stochastic optimal control problems is rooted in stochastic programming [39]. Stochastic MPC allows for constraint violation and aims to systematically balance improved performance and constraint satisfaction based on the probabilistic nature of the uncertainty. Typically, cost function is statistically evaluated in terms of expected value, while constraints are probabilistically described using chance constraints that allow for constraint violation as long as the probability of occurrence remains below a specified threshold. However, since underlying systems are often multidimensional and multiple constraints are imposed on the state or state trajectory, the evaluation of chance constraints involves multidimensional integrals, making it computationally challenging. The feasible set of chance constraints is generally nonconvex, further complicating chance-constrained stochastic optimal control problems. Moreover, propagating uncertainty through complex dynamics to obtain the distribution of predicted system trajectories (or the state distribution at each time stage) is difficult. All these reasons result in that in its current form, stochastic MPC is an exploding field, which is complicated and cluttered.

Safety is typically defined using a predefined set of states represented by a set of inequalities. Individual chance constraints and stage-wise chance constraints (also called joint chance constraints) are commonly employed in the stochastic MPC community [21]. Individual chance constraints provide a probabilistic safety guarantee for each inequality satisfaction of the state at each time stage, while stage-wise chance constraints ensure a probabilistic safety certificate for the simultaneous satisfaction of a whole set of inequalities for the state at each time stage. Compared to individual chance constraints, stage-wise chance constraints are more natural but more difficult to handle, as explained in [21, Section 2.2]. Thus, the focus of Chapters 4 and 5 is on the stochastic MPC with stage-wise chance constraints

In certain real-world scenarios, such as automated vehicles, robot path planning, and drone flights, the duration of the control task is typically finite. Any crashes during the mission can lead to severe damage to the vehicle, robot, or drone, resulting in mission failure. In this context, any state trajectory that deviates from the safe set during the mission is considered unsafe. The chance constraint naturally arises in terms of the state trajectory within the mission duration, so we refer to it as a mission-wide (or dynamically joint) chance constraint. The differences between mission-wide and stage-wise chance constraints are visualized in Figure 1.3 using an example of drone flights. Even in settings with infinite time duration, the mission-wide chance constraint is typically more meaningful, provided that the controlled state can reach a set within finite time steps and remain in this set for the remaining mission duration. Clearly, mission-wide chance constraints are even more complex than stage-wise chance constraints because the state trajectory contains more states that are coupled through the system dynamics.

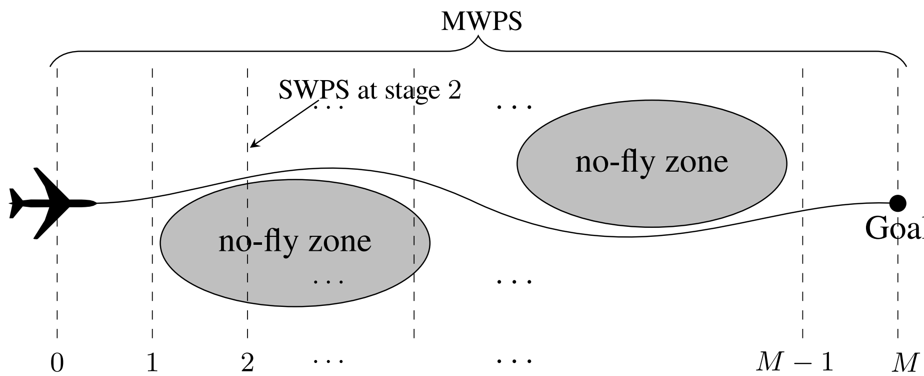

Figure 1.3: An example of drone flights. Mission-Wide Probability of Safety (MWPS) pertains to the entire mission duration, whereas Stage-Wise Probability of Safety (SWPS) applies to individual time stages.

In chapter 6, we argue that conventional dynamic programming method fails to directly apply to the mission-wide chance-constrained optimal control problems. To handle these new types of constraints, we propose a dynamic programming solution through proper state augmentation. However, this approach unfortunately comes with the challenge of working in functional spaces. Nonetheless, it can serve as a basis to inspire the development of further approximation methods. For instance, two preprints [40], [41] have emerged recently. In [40], the authors utilize a binary state augmentation, akin to our approach outlined in Chapter 6, to implement dynamic programming. Building upon this state augmentation, [41] introduces a linear programming approach to tackle mission-wide chance-constrained optimal control problems. Alternatively, in chapter 7, we propose a novel mission-wide safety-constrained stochastic MPC controller, implemented in a shrinking-time horizon fashion for linear systems, to approximate the mission-wide chance-constrained optimal control problems.

1.3 Contributions and Outline

This thesis consists of eight chapters. In the rest of this chapter, we elaborate on the contributions of this thesis in a chapter-by-chapter manner. Chapter 2 presents some preliminary theory essential for the subsequent chapters. The main contributions are introduced in Chapters 3–7. Chapter 8 concludes the thesis and discusses some future working directions. Figure 1.4 provides the outline of the thesis.

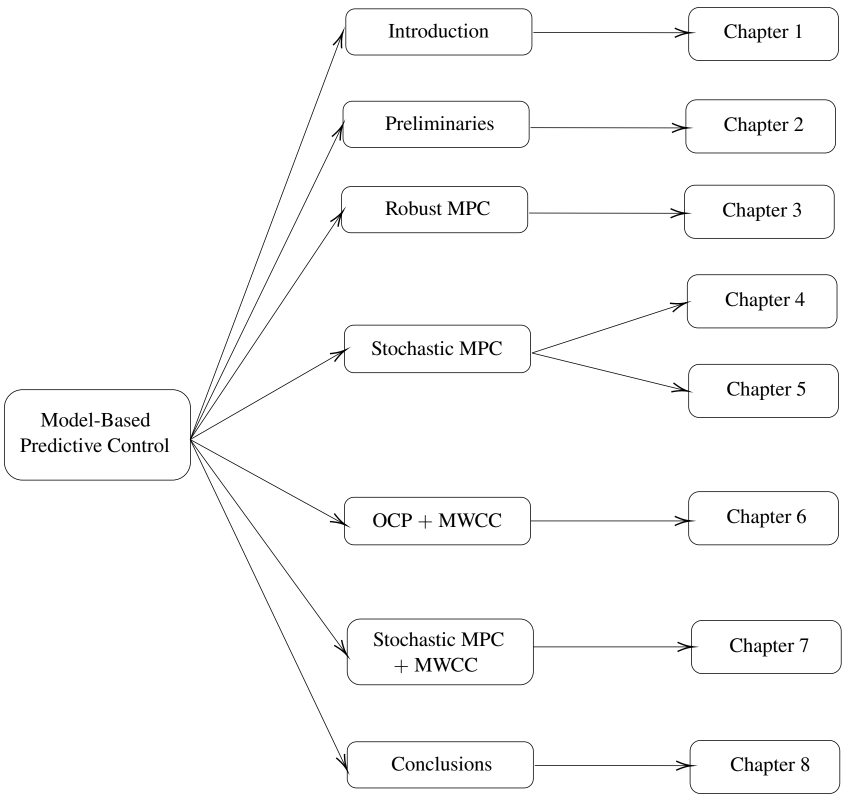

Figure 1.4: A brief overview of the thesis.(Abbreviations: Mission-Wide Chance Constraint (MWCC) and Optimal Control Problem (OCP))

Chapter 3: Tube MPC with Time-Varying Cross-Sections

Publication: Kai Wang, Sixing Zhang, Sébastien Gros and Saša V. Raković. Tube MPC with Time-Varying Cross-Sections. *Accepted to IEEE Transactions on Automatic Control*, 2024.

Contributions: This chapter considers tube model predictive control of discrete-time linear systems subject to additive bounded disturbances and mixed state and control constraints. An improved tube model predictive controller, leveraging the advantages and mitigating the disadvantages of three pivotal existing methods, is proposed. Its computational aspects and theoretical properties are thoroughly discussed and compared with its predecessor methods. Specifically, the refined tube MPC maintains all the favorable computational and structural characteristics of its predecessors, and it also enlarges the feasible domain and enhances the stability guarantees to some reasonable extent. Additionally, it streamlines the offline computations through the utilization of support functions.

Chapter 4: Robustifying MPC of Chance Constrained Linear Systems

Publication: Kai Wang, Kiet Tuan Hoang and Sébastien Gros. Robustifying Model Predictive Control of Uncertain Linear Systems with Chance Constraints. *Accepted to IEEE Conference on Decision and Control (CDC)*, 2024.

Contributions: This chapter proposes a model predictive controller for discrete-time linear systems with additive, possibly unbounded, stochastic disturbances and subject to chance constraints. By computing a polytopic probabilistic positively invariant set for constraint tightening with the help of the computation of the minimal robust positively invariant set, the chance constraints are guaranteed, assuming only the mean and covariance of the disturbance distribution are given. The resulting online optimization problem is a standard strictly quadratic programming, just like in conventional model predictive control with recursive feasibility and stability guarantees and is simple to implement.

Chapter 5: Stochastic Linear MPC with Sound Control-Theoretic Properties

Publication: Kai Wang, Oliver K. Hasler and Sébastien Gros. Stochastic MPC with Robust Positive Invariance and Exponential Stability Guarantees. *Submitted to IEEE Transactions on Automatic Control*, 2024.

Contributions: This chapter investigates stochastic MPC of linear systems in the presence of additive, bounded, stochastic disturbances and subject to joint state chance constraints and hard input constraints. As it is in the case of rigid tube MPC, the optimal control problem solved online, incorporates the initial condition of the nominal system as a decision variable. This incorporation is a key ingredient in establishing a strong stability result, namely robust exponential stability of a robust positively invariant set for the controlled dynamics. The mixed stochastic/worst-case tightening procedure is utilized to establish the positive invariance property of the controllable set. The resulting stochastic MPC algorithm solves online a quadratic programming problem and requires a similar offline computational effort to that required in classic tube MPC.

Chapter 6: Mission-Wide Chance-Constrained Optimal Control via DP

Publication: Kai Wang, and Sébastien Gros. Solving Mission-Wide Chance-Constrained Optimal Control Using Dynamic Programming. In *Proceedings of the 61st IEEE Conference on Decision and Control (CDC)*, 2022.

Contributions: This chapter aims to provide a Dynamic Programming (DP) approach to solve the Mission-Wide Chance-Constrained Optimal Control Problems (MWCC-OCP). The mission-wide chance constraint guarantees that the probability that the entire state trajectory lies within a constraint/safe region is higher than a prescribed level, and is different from the stage-wise chance constraints imposed at individual time steps. The control objective is to find an optimal policy sequence that achieves both (i) satisfaction of a mission-wide chance constraint, and (ii) minimization of a cost function. By transforming the stage-wise chance-constrained problem into an unconstrained counterpart via Lagrangian method, standard DP can then be deployed. Yet, for MWCC-OCP, this methods fails to apply, because the mission-wide chance constraint cannot be easily formulated using stage-wise chance constraints due to the time-correlation between the latter (individual states are coupled through the system dynamics). To fill this gap, firstly, we detail the conditions required for a classical DP solution to exist for this type of problem; secondly, we propose a DP solution to the MWCC-OCP through state augmentation by introducing an additional functional state variable.

Chapter 7: Stochastic MPC with Mission-Wide Chance Constraints

Publication: Kai Wang, and Sébastien Gros. Recursive Feasibility of Stochastic Model

Predictive Control with Mission-Wide Probabilistic Constraints. In *Proceedings of the 60th IEEE Conference on Decision and Control (CDC)*, 2021.

Contributions: This chapter is concerned with solving chance-constrained finite-horizon optimal control problems, with a particular focus on the recursive feasibility issue of stochastic MPC in terms of mission-wide probability of safety (MWPS). MWPS assesses the probability that the entire state trajectory lies within the constraint set, and the objective of the stochastic MPC controller is to ensure that it is no less than a threshold value. This differs from classic stochastic MPC where the probability that the state lies in the constraint set is enforced independently at each time instant. Unlike robust MPC, where strict recursive feasibility is satisfied by assuming that the uncertainty is supported by a compact set, the proposed concept of recursive feasibility for MWPS is based on the notion of remaining MWPSs, which is conserved in the expected value sense. We demonstrate the idea of mission-wide stochastic MPC linear stochastic systems by deploying a scenario-based algorithm.

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Chapter 2

Preliminaries

2.1 Formulation of Model Predictive Control

We consider the following discrete-time systems of the form

$$x^+ = f(x, u). \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the current state, $x \in \mathbb{R}^m$ is the current input and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the transition map that assigns the state $x^+ \in \mathbb{R}^n$ at the next time instant to each state and control pair (x, u). The function f is assumed to be continuous, and it satisfies

$$0 = f(0, 0),$$

i.e., the state and control pair $(0, 0)$ is the desired equilibrium. The following analysis is trivial to extended to the case where the desired equilibrium pair is specified by (x^s, u^s), i.e., $x^s = f(x^s, u^s)$.

The state x and control u are subject to the following hard constraints

$$(x, u) \in \mathcal{X} \times \mathcal{U}, \tag{2.2}$$

where the state constraint set $\mathcal{X} \subseteq \mathbb{R}^n$ is assumed to be closed, and the control constraint set $\mathcal{U} \subseteq \mathbb{R}^m$ is assumed to be compact. Both sets $\mathcal{X}$ and $\mathcal{U}$ are assumed to contain the origin in their interior.

For a state $x \in \mathbb{R}^n$, MPC solves the following finite-horizon, open-loop optimal control problem

$$
\begin{aligned}
V_N^0(x) = \min_{\mathbf{x}_N, \mathbf{u}_{N-1}} & \sum_{k=0}^{N-1} \ell(x_k, u_k) + V_f(x_N), && \text{(2.3a)}\\
\text{s.t.} \quad & x_{k+1} = f(x_k, u_k), \quad \forall k \in \{0, 1, \ldots, N-1\}, && \text{(2.3b)}\\
& (x_k, u_k) \in \mathcal{X} \times \mathcal{U}, \quad \forall k \in \{0, 1, \ldots, N-1\}, && \text{(2.3c)}\\
& x_0 = x, && \text{(2.3d)}\\
& x_N \in \mathcal{X}_f \subseteq \mathcal{X}. && \text{(2.3e)}
\end{aligned}
$$

The stage cost $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ and terminal cost $V_f : \mathcal{X}_f \to \mathbb{R}_{\geq 0}$ are assumed to be continuous, which typically penalize the distance to the desired equilibrium and origin respectively. Naturally, we assume that $\ell(0,0) = 0$ and $V_f(0) = 0$. Given a prediction horizon $N \in \mathbb{N}$, The decision variables are specified by $\mathbf{x}_N := (x_0^\top, x_1^\top, \ldots, x_N^\top)^\top$ and $\mathbf{u}_{N-1} := (u_0^\top, u_1^\top, \ldots, u_{N-1}^\top)^\top$.

Definition 2.1 *A set $\Omega \subseteq \mathcal{X} \subseteq \mathbb{R}^n$ is called positive invariant for the system $x^+ = f(x)$ with constraint $x \in \mathcal{X}$ if $x \in \Omega$ implies $f(x) \in \Omega$*

Definition 2.2 *A set $\Omega \subseteq \mathcal{X} \subseteq \mathbb{R}^n$ is called control positive invariant for the system $x^+ = f(x, u)$ with constraints $(x, u) \in \mathcal{X} \times \mathcal{U}$ if for all $x \in \Omega$, there exists a control $u \in \mathcal{U} \subseteq \mathbb{R}^m$ such that $f(x, u) \in \Omega$*

The terminal set $\mathcal{X}_f$ is compact and control invariant for the system $x^+ = f(x, u)$ and constraints $(x, u) \in \mathcal{X} \times \mathcal{U}$, and it contains the origin in its interior. This terminal set is the key ingredient in establishing recursive feasibility of the MPC controller (2.3). The set of admissible decision variables is given by

$$
\forall x \in \mathbb{R}^n, \quad \mathcal{D}_N(x) = \{(\mathbf{x}_N, \mathbf{u}_{N-1}) \,:\, \text{relations (2.3b)} - \text{(2.3e) hold}\}. \qquad (2.4)
$$

The feasible domain for the optimal control problem (2.3), which is also known as the N-step controllable set, is specified by

$$
\mathcal{C}_N = \{x \,:\, \mathcal{D}_N(x) \neq \emptyset\}.
$$

Within this setting, the optimal control problem (2.3) is guaranteed to be well-posed, and the value function $V_N^0(\cdot)$ exists for any state $x \in \mathcal{C}_N$ [1, Proposition 2.4]. We denote the optimal solutions of the optimal control problem (2.3) as

$$
\forall x \in \mathcal{C}_N, \quad \mathbf{x}_N^0(x) \text{ and } \mathbf{u}_{N-1}^0(x).
$$

The MPC controller evaluates implicitly the control law $\kappa_N : \mathcal{C}_N \mapsto \mathcal{U}$, specified by

$$\forall x \in \mathcal{C}_N, \quad \kappa_N(x) = u_0^0(x),$$

and it induces the model predictive controlled dynamics

$$\forall x \in \mathcal{C}_N, \quad x^+ = f(x, \kappa_N(x)). \tag{2.5}$$

Since MPC relies on the receding-horizon solution of the optimization problem (2.3), it is crucial to ensure that once (2.3) is solvable at the current state x, it remains solvable for the state x^+ at the next time instant, i.e., for all $x \in \mathcal{C}_N$, it holds that $x^+ \in \mathcal{C}_N$ with $x^+ = f(x, \kappa_N(x))$. This leads to the so-called recursive feasibility of MPC controllers, which is also known as the positive invariance of the set $\mathcal{C}_N$ for the model predictive controlled dynamics (2.5) and the implicitly induced constraints $(x, \kappa_N(x)) \in \mathcal{X} \times \mathcal{U}$. Within the above setting, the recursive feasibility is guaranteed.

In the following, we discuss the stability properties of the MPC controller. The main idea is to show that the value function V_N^0 is a valid Lyapunov function for the model predictive controlled dynamics (2.5).

Assumption 2.1 (Lower bound of $\ell(\cdot)$) *There exists a function $\gamma_1 \in \mathcal{K}_\infty$ [1] such that*

$$\forall x \in \mathcal{C}_N, \quad \forall u \in \mathcal{U}, \quad \gamma_1(\|x\|) \leq \ell(x, u). \tag{2.6}$$

Assumption 2.2 (Upper bound of $V_f(\cdot)$) *There exists a function $\gamma_2 \in \mathcal{K}_\infty$ such that*

$$\forall x \in \mathcal{X}_f, \quad V_f(x) \leq \gamma_2(\|x\|). \tag{2.7}$$

In addition, the terminal cost function $V_f(\cdot)$ satisfies local decrease condition

$$\forall x \in \mathcal{X}_f, \ \exists u \in \mathcal{U}, \quad V_f(f(x, u)) - V_f(x) \leq -\ell(x, u). \tag{2.8}$$

At this point, we are ready to recall the stability result as follows.

Theorem 2.1 (Asymptotic stability) *Suppose Assumptions 2.1 and 2.2 hold. There exist $\beta_1, \beta_2 \in \mathcal{K}_\infty$ such that for all $x \in \mathcal{C}_N$,*

$$\beta_1(\|x\|) \leq V_N^0(x) \leq \beta_2(\|x\|),$$
$$V_N^0(f(x, \kappa_N(x))) - V_N^0(x) \leq -\beta_1(\|x\|).$$

[1] A function belongs to class $\mathcal{K}_\infty$ if it is continuous, zero at the origin, strictly increasing, and unbounded.

The origin is asymptotically stable in $\mathcal{C}_N$ for the model predictive controlled dynamics (2.5).

Proof: The proof can be found in [1, Theorem 2.19]. □

Theorem 2.2 (Exponential stability) *Suppose Assumptions 2.1 and 2.2 hold in the case that the $\mathcal{K}_\infty$ functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ are replaced by functions $\alpha_1\|\cdot\|^\sigma$ and $\alpha_2\|\cdot\|^\sigma$ respectively, with $\alpha_1, \alpha_2, \sigma > 0$. Then, there exists $a_1, a_2, \sigma > 0$ such that for all $x \in \mathcal{C}_N$,*

$$\begin{aligned} a_1\|x\|^\sigma &\leq V_N^0(x) \leq a_2\|x\|^\sigma, \\ V_N^0(f(x,\kappa_N(x))) - V_N^0(x) &\leq -a_1\|x\|^\sigma. \end{aligned}$$

The origin is exponentionally stable in $\mathcal{C}_N$ for the model predictive controlled dynamics (2.5).

Proof: The proof can be found in [1, Theorem B.19]. □

Some variants of the MPC scheme introduced in this section, such as MPC without the terminal ingredients, MPC with zero terminal ingredients (i.e., $\mathcal{X}_f = \{0\}$ and $V_f(\cdot) \equiv 0$) and MPC with economic performance index (also known as economic MPC) can be found in the books [1], [2].

2.2 Formulation of Robust MPC

This section considers the uncertain discrete-time systems, described by

$$x^+ = f(x, u, w). \tag{2.9}$$

where $x \in \mathbb{R}^n$ represents the current state, $u \in \mathbb{R}^m$ stands for the current input, $w \in \mathbb{R}^s$ represents the uncertainty, and $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^s \to \mathbb{R}^n$ denotes the transition map. This map assigns the state $x^+ \in \mathbb{R}^n$ at the subsequent time instant for each combination of state, control, and uncertainty (x, u, w). Typically, the controller assumes that a control input u is applied at the current state x, while the uncertainty w remains unknown.

The system (2.9) are subject to the following hard constraints

$$(x, u, w) \in \mathcal{X} \times \mathcal{U} \times \mathcal{W}, \tag{2.10}$$

where $\mathcal{X} \subseteq \mathbb{R}^n$ is assumed to be closed, and $\mathcal{U} \subseteq \mathbb{R}^m$ and $\mathcal{W} \subseteq \mathbb{R}^s$ are assumed to be compact and contain the origin in their interior. Let us denote

$$\Xi := \{\mu \mid \mu : \mathbb{R}^n \to \mathbb{R}^m\}$$

as the set of control laws. The control policy associated with the underlying robust optimal control problem is a sequence of control laws, specified by

$$\boldsymbol{\mu}_{N-1} = \{\mu_0, \mu_1, \dots, \mu_{N-1}\} \tag{2.11}$$

for a given prediction horizon $N \in \mathbb{N}$, where $\mu_k \in \Xi$ for each relevant k.

For a state $x \in \mathbb{R}^n$, robust MPC proceeds by solving the following finite-horizon, min-max optimal control problem

$$\begin{aligned}
\min_{\boldsymbol{\mu}_{N-1}} \max_{\mathbf{w}_{N-1} \in \mathcal{W}^N} & \sum_{k=0}^{N-1} \ell^R(x_k, \mu_k(x_k), w_k) + V_f^R(x_N), && \text{(2.12a)}\\
\text{s.t.} \quad & x_{k+1} = f(x_k, \mu_k(x_k), w_k), \quad \forall k \in \{0, 1, \dots, N-1\}, && \text{(2.12b)}\\
& (x_k, \mu_k(x_k)) \in \mathcal{X} \times \mathcal{U}, \quad \forall k \in \{0, 1, \dots, N-1\}, && \text{(2.12c)}\\
& x_0 = x, && \text{(2.12d)}\\
& x_N \in \mathcal{X}_f \subseteq \mathcal{X}. && \text{(2.12e)}
\end{aligned}$$

Here, we denote the disturbance sequence as $\mathbf{w}_{N-1} := (w_0^\top, w_1^\top, \dots, w_{N-1}^\top)^\top$. The stage cost $\ell^R : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^s \to \mathbb{R}_{\geq 0}$ and terminal cost $V_f^R : \mathcal{X}_f \to \mathbb{R}_{\geq 0}$ are assumed to be continuous and positive definite.

Definition 2.3 *A set $\Omega \subseteq X$ is called robust positively invariant for the system $x^+ = f(x, w)$ and constraints $(x, w) \in X \times W \subseteq \mathbb{R}^n \times \mathbb{R}^s$ if and only if for all $x \in \Omega$ and all $w \in W$, it holds that $f(x, w) \in \Omega$.*

The terminal set $\mathcal{X}_f$ is assumed to be closed and robust positively invariant for the local dynamics $x^+ = f(x, \kappa_f(x), w)$, where $\kappa_f \in \Xi$, subject to constraints $(x, \kappa_f(x), w) \in \mathcal{X} \times \mathcal{U} \times \mathcal{W}$. The local control law $\kappa_f : \mathbb{R}^n \to \mathbb{R}^m$ is commonly determined offline. In addition, the corresponding maximal robust positively invariant set $\mathcal{X}_f$ is typically desired. The set of admissible control policies is given by

$$\forall x \in \mathbb{R}^n, \quad \Pi_N^R(x) = \{\boldsymbol{\mu}_{N-1} \, : \, \text{relations (2.12b)} - \text{(2.12e) hold}\}\,.$$

The effective domain for the min-max robust optimal control problem (2.12) is specified by

$$\mathcal{C}_N^R = \left\{x \,:\, \Pi_N^R(x) \neq \emptyset\right\}.$$

Within the above setting, the infinite–dimensional min-max optimization problem (2.12) is well posed. The optimal solutions of the infinite–dimensional min-max optimal control problem (2.12) are denoted by

$$\forall x \in \mathcal{C}_N^R, \quad \boldsymbol{\mu}_{N-1}^0(x).$$

The min-max robust MPC controller implicitly evaluates the control law $\kappa_N : \mathcal{C}_N^R \to \mathcal{U}$, defined by

$$\forall x \in \mathcal{C}_N^R, \quad \kappa_N^R(x) = \mu_0^0(x), \tag{2.13}$$

which leads to the closed-loop uncertain dynamics

$$\forall x \in \mathcal{C}_N^R, \quad x^+ = f(x, \kappa_N^R(x), w). \tag{2.14}$$

Analogous to Assumptions 2.1 and 2.2 in Section 2.1, stabilizing conditions are imposed on the stage and terminal cost functions, ℓ and V_f, to guarantee the stability of the resulting controlled uncertain dynamics (2.14).

Assumption 2.3

- *There exists a function $\gamma_1 \in \mathcal{K}_\infty$ such that*

$$\forall (x, u, w) \in \mathcal{X} \times \mathcal{U} \times \mathcal{W}, \quad \gamma_1(\|x\|) \leq \ell^R(x, u, w). \tag{2.15}$$

- *There exists a function $\gamma_2 \in \mathcal{K}_\infty$ such that*

$$\forall x \in \mathcal{X}_f, \quad V_f^R(x) \leq \gamma_2(\|x\|). \tag{2.16}$$

In addition, the terminal set $\mathcal{X}_f$ contains the origin in its interior, and the terminal cost function $V_f^R(\cdot)$ satisfies local decrease condition

$$\forall x \in \mathcal{X}_f, \quad V_f^R(f(x, \kappa_f(x), 0)) - V_f^R(x) \leq -\ell^R(x, \kappa_f(x), 0). \tag{2.17}$$

Theorem 2.3 *Suppose Assumption 2.3 hold. The robust model predictive controlled dynamics* (2.14) *is asymptotically stable with respect to a robust positively invariant set* *for dynamics* (2.14) *and constraints* (2.10). *If the condition* (2.17) *in Assumption 2.3 is replaced by*

$$\forall x \in \mathcal{X}_f,\ \forall w \in \mathcal{W}, \quad V_f^R(f(x, \kappa_f(x), w)) - V_f^R(x) \leq -\ell^R(x, \kappa_f(x), w), \quad (2.18)$$

then the origin is asymptotically stable in $\mathcal{C}_N^R$ *for the min-max robust model predictive controlled dynamics* (2.14).

Proof: The detailed proof and discussion can be found in [1, Chapter 3]. □

2.2.1 Tube-Based Robust MPC Formulation

This subsection is mainly reviewed from a collection of articles [3]–[7].

We denote the closed-loop set propagation function as

$$f(X, \mu) = \{f(x, \mu(x), w) \,:\, x \in X,\ w \in W\}\,, \quad (2.19)$$

for all sets $X \in \mathbb{R}^n$ and all control laws $\mu \in \Xi$. Then, for a state $x \in \mathbb{R}^n$, a tube-based robust optimal control problem is given, with $\mathbf{X}_N = (X_0, X_1, \ldots, X_N)$, by

$$\begin{aligned}
\min_{\mathbf{X}_N,\, \boldsymbol{\mu}_{N-1}} \quad & \sum_{k=0}^{N-1} L(X_k, \mu_k) + L_f(X_N). && (2.20a)\\
\text{s.t.} \quad & X_{k+1} \supseteq f(X_k, \mu_k), \quad \forall k \in \{0, 1, \ldots, N-1\}, && (2.20b)\\
& X_k \subseteq \mathcal{X},\ \mu_k \in \Xi, \quad \forall k \in \{0, 1, \ldots, N-1\}, && (2.20c)\\
& x \in X_0, && (2.20d)\\
& X_N \subseteq \mathcal{X}_f \subseteq \mathcal{X}. && (2.20e)
\end{aligned}$$

Here, the stage and terminal cost functions are defined as $L : \mathbb{C}^n \times \Xi \to \mathbb{R}$ and $L_f : \mathbb{C}^n \to \mathbb{R}$, where $\mathbb{C}^n$ denotes the set of compact sets in $\mathbb{R}^n$, and they are assumed to be continuous. The terminal constraint set $\mathcal{X}_f$ is also assumed to be closed and robust positively invariant for the local dynamics $x^+ = f(x, \kappa_f(x), w)$ and constraints $(x, \kappa_f(x), w) \in \mathcal{X} \times \mathcal{U} \times \mathcal{W}$, i.e., it satisfies the following set-algebraic condition

$$f(\mathcal{X}_f, \kappa_f) \subseteq \mathcal{X}_f \subseteq \mathcal{X}, \text{ with } \kappa_f \in \Xi.$$

Within the above setting, the optimization problem (2.20) is well-posed such that there is a solution exists for a state x in its feasible set [5]. In addition, the tube MPC controller resulting from (2.20) is recursive feasible.

The set of admissible control policies for (2.20) is given by

$$\forall x \in \mathbb{R}^n, \quad \Pi_N^T(x) = \{\boldsymbol{\mu}_{N-1} : \text{relations (2.20b)} - \text{(2.20e) hold}\}.$$

The effective domain for the tube optimal optimal control problem (2.20) is specified by

$$\mathcal{C}_N^T = \left\{x : \Pi_N^T(x) \neq \emptyset\right\}.$$

Now, we denote the optimal solutions of the tube optimal control problem (2.20) by

$$\forall x \in \mathcal{C}_N^T, \quad \mathbf{X}_N^0(x) \text{ and } \boldsymbol{\mu}_{N-1}^0(x).$$

Using the optimal solution $\boldsymbol{\mu}_{N-1}^0(\cdot)$, one defines the tube MPC policy $\kappa_N^T : \mathcal{C}_N^T \to \mathcal{U}$ as

$$\forall x \in \mathcal{C}_N^T, \quad \kappa_N^T(x) = \mu_0^0(x),$$

such that the closed-loop tube model predictive controlled uncertain dynamics is specified by

$$\forall x \in \mathcal{C}_N^T, \quad x^+ = f(x, \kappa_N^T(x), w). \tag{2.21}$$

The resulting tube model predictive controlled uncertain system is guaranteed to be asymptotically stable with mild assumptions given in [5].

In the rest of this section, we delve into a specific scenario where the tube model predictive control system is ensured to stabilize towards a minimal robust positively invariant set. Let $\mathcal{X}_{\mathcal{O}}$ be the minimal robust positive invariant set for the the local dynamics $x^+ = f(x, \kappa_f(x), w)$ and constraints $(x, \kappa_f(x), w) \in \mathcal{X} \times \mathcal{U} \times \mathcal{W}$. This set can be alternatively characterized by the following set-equation

$$f(\mathcal{X}_{\mathcal{O}}, \kappa_f) = \mathcal{X}_{\mathcal{O}}.$$

The conditions determining the existence and uniqueness of the minimal robust positively invariant set $\mathcal{X}_{\mathcal{O}}$ are detailed in [8].

Because the computational cost of (2.20) is usually prohibitive, it desirable for one to obtain a computational tractable tube-based model predictive control. One popular way is to parametrize the tube $\mathbf{X}_N$. Suppose that a sequence of nominal states $\mathbf{z}_N := (z_0^\top, z_1^\top, \ldots, z_N^\top)^\top$, generated by the nominal system $z^+ = f(z, \mu(z), 0)$, with $\mu \in \Xi$, satisfies the condition

$$\forall k \in \{0, 1, \ldots, N\}, \quad z_k \in X_k.$$

If $\mathbf{z}_N$ is used as part of the parameters of the tube $\mathbf{X}_N$, we can replaces the stage and terminal functions, L and L_f, by

$$\widetilde{\ell} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0} \quad \text{and} \quad \widetilde{V}_f : \mathbb{R}^n \to \mathbb{R}_{\geq 0},$$

respectively. Additionally, $\widetilde{\ell}$ and $\widetilde{V}_f$ are assumed to be continuous. Now, the tube optimal control problem, with parameterized nominal states, is described as

$$\begin{aligned} V_N^0(x) = \min_{\mathbf{X}_N, \boldsymbol{\mu}_{N-1}} & \sum_{k=0}^{N-1} \widetilde{\ell}(z_k, \mu_k(z_k)) + \widetilde{V}_f(z_N), && \text{(2.22a)} \\ \text{s.t. } & z_{k+1} = f(z_k, \mu_k(z_k), 0), \quad \forall k \in \{0, 1, \ldots, N-1\}, && \text{(2.22b)} \\ & X_{k+1} \supseteq f(X_k, \mu_k), \quad \forall k \in \{0, 1, \ldots, N-1\}, && \text{(2.22c)} \\ & z_k \in X_k \subseteq \mathcal{X},\ \mu_k \in \Xi, \quad \forall k \in \{0, 1, \ldots, N-1\}, && \text{(2.22d)} \\ & x - z_0 \in X_0, && \text{(2.22e)} \\ & X_N \subseteq \mathcal{X}_f \subseteq \mathcal{X}. && \text{(2.22f)} \end{aligned}$$

Here, note that $\mathbf{z}_N$ is partial components of the underlying parameters that represent the tube $\mathbf{X}_N$. Within the above setting, this optimization problem is well-formulated. Now, we denote the set of admissible control policies for (2.22) by

$$\forall x \in \mathbb{R}^n, \quad \widetilde{\Pi}_N(x) = \{(\mathbf{X}_N, \boldsymbol{\mu}_{N-1}) \ : \ \text{relations (2.22b)} - \text{(2.22f) hold}\}.$$

The effective domain for (2.22) is specified by

$$\widetilde{\mathcal{C}}_N = \left\{x \ : \ \widetilde{\Pi}_N(x) \neq \emptyset\right\}.$$

Now, we denote the optimal solutions of the parameterized tube optimal control problem (2.22) by

$$\forall x \in \widetilde{\mathcal{C}}_N, \quad \mathbf{X}_N^0(x) \text{ and } \boldsymbol{\mu}_{N-1}^0(x).$$

Using the optimal solution $\boldsymbol{\mu}_{N-1}^0(\cdot)$, one defines the resulting closed-loop policy as

$$\forall x \in \widetilde{\mathcal{C}}_N, \quad \widetilde{\kappa}_N(x) = \mu_0^0(x),$$

so that the closed-loop tube model predictive controlled uncertain dynamics

$$\forall x \in \widetilde{\mathcal{C}}_N, \quad x^+ = f(x, \widetilde{\kappa}_N(x), w). \tag{2.23}$$

Let us define $|x|_A := \inf_{a \in \mathcal{A}} \|x - a\|$ as the distance of a point $x \in \mathbb{R}^n$ from a set $\mathcal{A} \subseteq \mathbb{R}^n$. We make uses of the following assumption.

Assumption 2.4

- *The stage cost $\tilde{\ell}$ is equal to zero when $x \in \mathcal{X}_{\mathcal{O}}$ and the local terminal control law κ_f is applied, i.e.,*
$$\forall x \in \mathcal{X}_{\mathcal{O}}, \quad \tilde{\ell}(x, \kappa_f(x)) = 0.$$
- *There exists a function $\gamma_1 \in \mathcal{K}_\infty$ such that*
$$\forall (x, u) \in \mathcal{X} \times \mathcal{U}, \quad \gamma_1(\|x\|_{\mathcal{X}_{\mathcal{O}}}) \leq \tilde{\ell}(x, u). \tag{2.24}$$
- *There exists a function $\gamma_2 \in \mathcal{K}_\infty$ such that*
$$\forall x \in \mathcal{X}_f, \quad \widetilde{V}_f(x) \leq \gamma_2(\|x\|). \tag{2.25}$$
In addition, the terminal cost function $\widetilde{V}_f(\cdot)$ satisfies the local decrease condition
$$\forall x \in \mathcal{X}_f, \, \forall w \in \mathcal{W}, \quad \widetilde{V}_f(f(x, \kappa_f(x), w)) - \widetilde{V}_f(x) \leq -\tilde{\ell}(x, \kappa_f(x)). \tag{2.26}$$

Theorem 2.4 (Robust asymptotic stability and exponential stability) *Suppose Assumption 2.4 hold. The minimal robust positively invariant set $\mathcal{X}_{\mathcal{O}}$ is robust asymptotically stable for the uncertain dynamics* (2.23) *with a region of attraction being equal to the set $\widetilde{\mathcal{C}}_N$. Furthermore, if the $\mathcal{K}_\infty$ functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ in Assumption 2.4 are replaced by functions $\alpha_1\|\cdot\|^\sigma$ and $\alpha_2\|\cdot\|^\sigma$ respectively, with $\alpha_1, \alpha_2, \sigma > 0$, then the minimal robust positively invariant set $\mathcal{X}_{\mathcal{O}}$ is robust exponentially stable for the uncertain dynamics* (2.23) *with a region of attraction being equal to the set $\widetilde{\mathcal{C}}_N$.*

Proof: The proof can be found in [1], [3]. □

2.3 Formulation of Stochastic MPC

When the distribution of stochastic system uncertainty is well-characterized, it is more intuitive to leverage stochastic MPC. This approach exploits stochasticity to alleviate the conservatism associated with robust MPC and can even handle unbounded stochastic uncertainty.

In stochastic MPC, similar to robust MPC, the system to be controlled is typically described by the following stochastic system

$$x^+ = f(x, u, w), \text{ with } w \sim \omega,$$

in which ω denotes a probability distribution. The support set of the probability distribution ω is given by $\mathcal{W} \subseteq \mathbb{R}^s$, i.e., we have

$$\omega(\mathcal{W}) = 1.$$

In contrast to robust MPC, where we assume $\mathcal{W}$ is compact, in the context of stochastic MPC, the set $\mathcal{W}$ can be unbounded.

Similar to robust MPC, for a given prediction horizon $N \in \mathbb{N}$, the decision variable is typically assumed to be a policy

$$\boldsymbol{\mu}_{N-1} = \{\mu_0, \mu_1, \ldots, \mu_{N-1}\}, \tag{2.27}$$

where $\mu_k \in \Xi$ for each relevant k, and $\Xi := \{\mu \mid \mu : \mathbb{R}^n \to \mathbb{R}^m\}$ is defined as the set of control laws mapping a state x to a control input u.

The cost most commonly minimized online is defined as follows:

$$J_N(x, \boldsymbol{\mu}_{N-1}) = \mathbb{E}\left[\sum_{k=0}^{N-1} l(x_k, \mu_k(x_k)) + l_f(x_N) \,\middle|\, x_0 = x\right], \tag{2.28}$$

where x_k denotes the predicted states given the current state $x_0 = x$, control laws control policy $\boldsymbol{\mu}_{N-1}$ and disturbance sequence $\mathbf{w}_{N-1} = \{w_0, w_1, \ldots, w_{N-1}\}$, i.e., for all $k \in \{0, 1, \ldots, N-1\}, x_{k+1} = f(x_k, \mu_k(x_k), w_k)$ with $x_0 = x$. The functions $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ and $V_f : \mathcal{X}_f \to \mathbb{R}_{\geq 0}$ are know as the stage and terminal cost, respectively. The expectation $\mathbb{E}(\cdot)$ evaluated with respect to the probability distribution of the underlying probability space. Other cost functions are also occasionally used in the community, such as nominal certainty-equivalent cost and sample average cost.

Naturally, the minimization of the cost function (2.28) is performed while adhering to constraints on system states and/or inputs to ensure a degree of safety for the controlled system. While ensuring satisfaction of the hard constraints $x \in \mathcal{X}$ and $u \in \mathcal{U}$, as in robust MPC, is desirable, these constraints are often deemed overly conservative or even unnecessary in certain application scenarios. The most commonly investigated constraints in stochastic MPC is the so-called chance constraints, which guarantees that the probability of constraint violation is less than a specified value. Suppose the set of safe states $\mathcal{X}$ is represented as follows

$$\mathcal{X} = \{x \in \mathbb{R}^n \,:\, \forall i \in \mathcal{I},\ g_i(x) \leq 1\}, \tag{2.29}$$

where $g_i : \mathbb{R}^n \to \mathbb{R}$ is a measurable function for each $i \in \mathcal{I}$, with $\mathcal{I} := \{1, 2, \ldots, n_{\mathcal{X}}\}$ being a finite index set.

The stage-wise chance constraints are described as

$$\Pr(x \in \mathcal{X}) = \Pr(\, g_i(x) \leq 1, \text{ for all } i \in \mathcal{I}) \geq p, \tag{2.30}$$

where $p \in (0,1)$ is a modeling parameter. The notation $\Pr(\cdot)$ is the probability measure of the underlying probability space. The constraint (2.30) ensures that the state x must remain within the set $\mathcal{X}$ with a probability not less than the designated confidence level p.

The individual chance constraints, which are very commonly used in stochastic MPC literature, are of the form

$$\Pr(\, g_i(x) \leq 1) \geq p_i, \quad \forall i \in \mathcal{I}, \tag{2.31}$$

where for all $i \in \mathcal{I},\ p_i \in (0,1)$ are the designated confidence levels. These constraints only ensure satisfaction with respect to the constraint function g_i independently, which is less meaningful compared to stage-wise chance constraints (2.30), The reason is that in many application scenarios, we aim for the entire group of constraints representing the safe set $\mathcal{X}$ to be satisfied simultaneously. The popularity of individual chance constraints is very likely due to the fact that some type of these constraints can be accounted for in an analytical way or some other tractable ways. In a very conservative manner, one can make use of the individual chance constraints (2.31) to approximate stage-wise chance constraint (2.30) by choosing $\sum_{i \in \mathcal{I}} p_i \geq p - 1 + N$.

As already pointed in the introduction, this thesis is also concerned with the mission-wide chance constraint

$$\Pr\left(x_k \in \mathcal{X}, \text{ for all } k \in \{1, 2, \ldots, \bar{N}\}\right) \geq q, \tag{2.32}$$

where $\bar{N} \in \mathbb{N}$ denotes mission duration (or the length of the control task) and parameter $q \in (0,1)$ is the designated confidence level. Unlike the state-wise chance constraint (2.30), which enforces safety certificate on the single stage, the safety certificate provided by the mission-wide chance constraint is on the state trajectory. The the safety certificate on the state trajectory provided by the state-wise chance constraint is $p^{\bar{N}}$, which tends to zero when the mission duration $\bar{N}$ is large. The intuitive reason behind this issue is that stage-wise chance constraints neglect couplings over the states at different stages resulting from the system's dynamics. A more detailed commons are presented in Section 7.2.2.

Remark 2.1 *Other types of constraints defined on the random state x are also encountered in the literature, One of these studied constraints is expectation constraint.*

Suppose the probability distribution of the random state x is given by ρ, i.e., $x \sim \rho$. The stage-wise chance constraint on x can also be defined as follows

$$g(\rho) := \Pr(x \in \mathcal{X}) := \int_{\mathcal{X}} 1 \, \mathrm{d}\rho := \int_X \mathbf{1}_{\mathcal{X}}(x) \, \rho(\mathrm{d}x) := \rho(\mathcal{X}) \geq p$$

where $\mathbf{1}_{\mathcal{X}}$ denotes the indicator function of some set $\mathcal{X}$, defined by $\mathbf{1}_{\mathcal{X}}(x) = 1$ if $x \in \mathcal{X}$ and 0 otherwise. The expectation constraint is formulated as follows

$$g(\rho) := \mathbb{E}[x] := \int c \, \mathrm{d}\rho := \int_{\mathcal{X}} c(x) \, \rho(\mathrm{d}x) \geq p^e.$$

where the measurable function $c : \mathcal{X} \to \mathbb{R}$ is a constraint function of the state x so that we would like to keep its average value no worse than $p^e \in \mathbb{R}$. Notice that the stage-wise chance constraint is a special case of this expectation constraint if we set $c = \mathbf{1}_{\mathcal{X}}$ and $p = p^e$. The other forms of chance constraints can be similarly formulated as a special case of the expectation cost as well.

When the cost function (2.28) and stage-wise chance constraint (2.10) are considered, the stochastic optimal control problem, with prediction horizon $N \in \mathbb{R}$, can be formulated as follows

$$\min_{\boldsymbol{\mu}_{N-1}} \; \mathbb{E}\left[\sum_{k=0}^{N-1} l(x_k, \mu_k(x_k)) + l_f(x_N)\right] \tag{2.33a}$$

$$\text{s.t.} \quad x_{k+1} = f(x_k, \mu_k(x_k), w_k), \quad w_k \sim \omega, \quad \forall k \in \{0, 1, \ldots, N-1\}, \tag{2.33b}$$

$$\Pr(x_k \in \mathcal{X}) \geq p_k, \quad \forall k \in \{1, \ldots, N\}, \tag{2.33c}$$

$$\mu_k(x_k) \in \mathcal{U}, \quad \forall k \in \{0, \ldots, N-1\}, \tag{2.33d}$$

$$x_0 = x, \tag{2.33e}$$

where x is the current state, and $p_k \in (0, 1)$ for each relevant k. In some application scenarios, if appropriate, the hard input constraints (2.33d) can be replaced by the chance constraints

$$\Pr(\mu_k(x_k) \in \mathcal{U}) \geq p_k^u, \quad \forall k \in \{0, \ldots, N-1\},$$

with $p_k^u \in (0, 1)$ for each relevant k. Stochastic MPC proceeds by repeatedly solving the optimization problem (2.33) at each time step and applying the first optimized control law to the current state.

Computational aspects The decision variable $\boldsymbol{\mu}$ represents a control policy, residing in an infinite-dimensional function space. Thus, $\boldsymbol{\mu}$ is usually parameterized to simplify optimization. Despite this, the computation of the expected cost (2.33a) and the chance constraints (2.33c) can still be intractable. In practice, one usually either reformulates them into more tractable forms in a conservative manner or relies on assumptions about the systems (e.g., linear systems) and uncertainties (e.g., Gaussian disturbances).

Recursive feasibility, stability and convergences Roughly specking, there is no satisfying, unified framework to ensure these control-theoretic properties. For recursive feasibility, one popular way is to combines robust constraints with the chance constraints of the controlled system by assuming that the disturbance support set $\mathcal{W}$ is bounded. Another group of methods for ensuring recursive feasibility relies on the support of the associated nominal controlled systems. A rigorous treatment of stabilizing conditions stems from the theory of stability of discrete-time, controlled stochastic processes [9]. Various Convergences results can be found in the literature such as almost sure and in probability convergences of value function or closed-loop states. The relevant review of the relevant work can be found in Chapter 4 and 5. A more detailed review can be found in, for instance, [1, Section 3.7] [10, Chapter 7] and [11].

2.3.1 Mission-Wide Chance-Constrained Optimal Control

As argued in the above and Section 1.2, mission-wide chance constraints is, in general, more meaningful in defining safety on the system. In this subsection, we replace the stage-wise chance constraints in (2.33) with mission-wide chance constraint and remove the hard input constraints in (2.33). The resulting mission-wide chance-constrained stochastic optimal control problem is specified by

$$\min_{\boldsymbol{\mu}_{N-1}} \ \mathbb{E}\left[\sum_{k=0}^{\bar{N}-1} l(x_k, \mu_k(x_k)) + l_f(x_N)\right], \tag{2.34a}$$

$$\text{s.t.} \quad x_{k+1} = f(x_k, \mu_k(x_k), w_k), \quad w_k \sim \omega, \quad \forall k \in \{0, 1, \dots, \bar{N}-1\}, \tag{2.34b}$$

$$\Pr\left(x_k \in \mathcal{X}, \text{ for all } k \in \{1, \dots, \bar{N}\}\right) \geq p \tag{2.34c}$$

$$x_0 = x. \tag{2.34d}$$

We will tackle this problem precisely using dynamic programming in Chapter 6, and explore approximate solutions through stochastic MPC in Chapter 7. Problem (2.34) has also been explored within the realm of dynamic consistency and risk measures in multistage stochastic programming, as seen in works such as [12]–[14], and it has been approximately solved using reinforcement leaning [15]. Similar investigations have also been reported in the field of constrained Markov Decision Processes, particularly concerning expectation constraints [16], [17].

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Chapter 3

Tube MPC with Time-Varying Cross-Sections

This Chapter is based on the following paper under review:

Kai Wang, Sixing Zhang, Sébastien Gros and Saša V. Raković. Tube MPC with Time-Varying Cross-Sections. *Accepted to IEEE Transactions on Automatic Control*, 2024.

3.1 Introduction

Model Predictive Control (MPC) is a widely used, modern, optimization-based control technique [1], [2]. Robust MPC is an improved form of MPC, inherently designed to maintain stability, performance and constraint satisfication even in the face of uncertainty [3]. The range of robust MPC proposals includes open-loop minimax robust MPC [4], [5], theoretically complete and computationally intensive closed-loop robust MPC [6], [7], robust MPC based on dynamic programming [2], [8], disturbance affine feedback MPC [9], [10] and tube MPC [11]–[19] offering theoretically flexible and computationally tractable methods.

As pointed out in the review articles [3], [20], tube MPC has emerged as a leading paradigm for robust MPC due to its intuitive ease, computational simplicity and guaranteed control-theoretic properties. Based on the early tube MPC proposals [11], [12], a sequence of articles appeared over the last two decades, which has defined the state-of-the-art of tube MPC. The developed tube MPC methods include the so-called rigid tube MPC [13], homothetic tube MPC [15], [16], elastic tube MPC [18], parameterized tube MPC [17] and configuration-constrained tube MPC [21] for linear systems. Extensions of tube MPC for nonlinear systems have been investigated [22]–[25]. Tube MPC is also utilized within the contexts of robust output feedback MPC synthesis [26]–[28] and safe learning based control methods [29], [30]. More recently, the rigid tube MPC using implicit representations of terminal set and tube cross-section sets was also proposed in [14], which is applicable to potentially very high-dimensional systems.

The primary focus of this chapter is to refine the early and simple, but yet computationally effective, tube MPC methods [11]–[13] for linear systems with bounded disturbances. The tube MPC proposals [11]–[13] are effectively concerned with the same problem and have in common both their computational simplicity and strong control-theoretic properties. These methods are computationally efficient since they reduce to a conventional MPC when applied to a deterministic nominal system subject to modified stage constraints, appropriate terminal constraints, and disturbace-free stage and terminal cost functions. These methods also provide guarantees of robust convergence/stability and robust positive invariance. At the conceptual level, all these methods employ a rather simple state and control parameterization and an affine control policy in solving the tube optimal control problems. On the other hand, proposals [11]–[13] differ in terms of local uncertainty propagation, nominal state initialization and modified constraints as well as utilized cost functions. Roughly speaking, these differences lead to different features of these proposals, and each proposal has its own advantages and disadvantages, as discussed in more details in Section 3.5.1.

In this chapter, we revisit the early tube MPC methods [11]–[13] and report a refinement of these methods. The mixed state and control constraints are considered, which are more general than the separate state and control constraints specified in [11]–[13]. A novel state decomposition strategy, that further decomposes the local uncertainty into two components, is proposed. Based on this new decomposition strategy, we propose a tube MPC method that improves both robust convergence of [11] and robust asymptotic stability of [12] to robust exponential stability, and it enlarges the effective domains of [12], [13] to a robust positively invariant set that is identical with the effective domain of [11] by construction. Moreover, based on the idea of [14], the support function is employed in constraint tightening to avoid the computationally expensive, explicit construction of tube cross-section sets. Due to the employment of support functions, the disturbance set, which contains the origin in its interior, need not be polytopic; it merely requires being convex and compact.

The rest of this chapter is organized as follows. Section 3.2 presents the problem setup and the objectives of this chapter. Section 3.3 describes the constraints on the employed tubes and specifies the set of admissible decision variables, and it gives the utilized cost functions. Section 3.4 proposes the improved tube MPC and analyzes its control-theoretic properties. Section 3.5 provides a detailed comparison and offers two numerical illustrations. Section 3.6 ends this chapter with concluding remarks.

3.1.1 Basic Nomenclature and Conventions

The sets of real numbers, non-negative integers and positive integers are denoted by $\mathbb{R}$, $\mathbb{N}$, respectively. Given $a, b \in \mathbb{N}$, with $a < b$, we use the notation $\mathbb{N}_{[a:b]}$ to denote the set of non-negative integers $\{a, a+1, \ldots, b\}$. We denote $\mathbb{N}_{[0:b]}$ by $\mathbb{N}_b$. Given with $p \in \mathbb{N}_+ \cup \{\infty\}$, the p-norm of a vector $x \in \mathbb{R}^n$ is denoted by $\|x\|_p$, while $\mathcal{B}_p^n$ denotes the corresponding closed unit p-norm ball. For convenience, we denote the Euclidean norm $\|\cdot\|_2$ by $\|\cdot\|$. The Minkowski sum and Pontryagin difference of nonempty sets $\mathcal{X}$ and $\mathcal{Y}$ in $\mathbb{R}^n$ are

$$\begin{aligned}\mathcal{X} \oplus \mathcal{Y} &:= \{x + y \,:\, x \in \mathcal{X},\ y \in \mathcal{Y}\} \text{ and} \\ \mathcal{X} \ominus \mathcal{Y} &:= \{x \,:\, \forall y \in \mathcal{Y},\ x + y \in \mathcal{X}\},\end{aligned}$$

respectively. The image of a nonempty set $\mathcal{X}$ under a matrix of compatible dimensions M is given by $M\mathcal{X} := \{Mx \,:\, x \in \mathcal{X}\}$. Likewise, if M is a square matrix, for any integer $k \in \mathbb{N}$, $M^k\mathcal{X} := \left\{M^k x \,:\, x \in \mathcal{X}\right\}$. A proper D–set in $\mathbb{R}^n$ is a closed convex subset of $\mathbb{R}^n$ that contains the origin in its interior. A proper C–set in $\mathbb{R}^n$ is a bounded proper D–set in $\mathbb{R}^n$. The intersection of finitely many closed half–spaces is a polyhedral set. A polytopic set is a bounded polyhedral set. The support function $\mathrm{h}(\mathcal{X}, \cdot)$ of a nonempty, closed, convex set $\mathcal{X} \subseteq \mathbb{R}^n$ is given, for all $y \in \mathbb{R}^n$, by

$$\mathrm{h}(\mathcal{X}, y) := \sup_x \{y^\top x \,:\, x \in \mathcal{X}\}.$$

Finally, we do not distinguish row vectors from column vectors unless necessary.

3.2 Problem Setup

3.2.1 System and Constraints

We consider discrete-time linear, time-invariant, uncertain systems given by

$$x^+ = Ax + Bu + w, \tag{3.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $w \in \mathbb{R}^n$ are, respectively, the current state, control and disturbance, and $x^+ \in \mathbb{R}^n$ denotes the successor state. The matrix pair $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ is *a priori* given and constant. At any current time, the state x is known, while the disturbance w is unknown but obeys the constraint

$$w \in \mathcal{W}. \tag{3.2}$$

Throughout this chapter, we consider mixed polyhedral state and control constraints

$$(x, u) \in \mathcal{Y} \quad \text{with} \quad \mathcal{Y} := \left\{(x, u) \ : \ \forall i \in \mathcal{I}_{\mathcal{Y}},\ c_i^\top x + d_i^\top u \leq 1\right\}. \tag{3.3}$$

We make the following standing assumptions on the system (3.1) and constraints (3.2)–(3.3).

Assumption 3.1

- *The matrix pair $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ is known, and it is strictly stabilizable.*
- *The disturbance constraint set $\mathcal{W} \subseteq \mathbb{R}^n$ is a proper C–set.*
- *The index set $\mathcal{I}_{\mathcal{Y}} := \{1, 2, \ldots, n_{\mathcal{Y}}\}$, with $n_{\mathcal{Y}} \in \mathbb{N}_+$, is finite, and for all $i \in \mathcal{I}_{\mathcal{Y}}$, $(c_i, d_i) \in \mathbb{R}^{n+m}$ are known. The set $\mathcal{Y}$ is a polyhedral proper D–set, and its representation are irreducible.*

3.2.2 State Decomposition

In this chapter, the parameterized state and control predictions and an affine control policy are employed. For a state $x \in \mathbb{R}^n$, a control policy $\Pi_{N-1}(\cdot)$ is a sequence of affine control laws $\{\pi_k(\cdot, \cdot, \cdot, \cdot)\}_{k\in\mathbb{N}_{N-1}}$, each term of which is parametrized via points $x_k \in \mathbb{R}^n$, $z_k \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^m$ and an *a priori* specified control matrix $K \in \mathbb{R}^{m\times n}$ as follows

$$\forall k \in \mathbb{N}_{N-1}, \quad \pi_k(x_k, z_k, v_k, x) := v_k + K(x_k - z_k), \tag{3.4}$$

so that

$$\forall k \in \mathbb{N}_{N-1}, \quad x_{k+1} = Ax_k + B(v_k + K(x_k - z_k)) + w_k, \tag{3.5}$$

starting from $x_0 = x$.

Assumption 3.2 *The matrix $K \in \mathbb{R}^{m\times n}$ is known, and it is such that the matrix $A_K := A + BK$ is strictly stable.*

The sequences $\{z_k\}_{k\in\mathbb{N}_N}$ and $\{v_k\}_{k\in\mathbb{N}_{N-1}}$ satisfy the following nominal dynamics

$$\forall k \in \mathbb{N}_{N-1}, \quad z_{k+1} = Az_k + Bv_k. \tag{3.6}$$

The predicted states are decomposed into three components

$$\forall k \in \mathbb{N}_N, \quad x_k = z_k + \bar{s}_k + \tilde{s}_k. \tag{3.7}$$

The nominal component z_k follows from the nominal controlled dynamics (3.6), and the local uncertain components $\bar{s}_k$ and $\tilde{s}_k$ are specified by the following two dynamics

$$\forall k \in \mathbb{N}_{N-1}, \quad \bar{s}_{k+1} = A_K\bar{s}_k \quad \text{with} \quad \bar{s}_0 = x - z_0, \quad \text{and} \tag{3.8}$$

$$\forall k \in \mathbb{N}_{N-1}, \quad \tilde{s}_{k+1} = A_K\tilde{s}_k + w_k \quad \text{with} \quad \tilde{s}_0 = 0. \tag{3.9}$$

The initial local uncertainty $\bar{s}_0$, which is a decision variable to be optimized, represents the deviation of the current state x from the initial nominal state z_0. Note that the component $\bar{s}_k$ evolves through the *deterministic* dynamics (3.8) starting from $\bar{s}_0$, while the component $\tilde{s}_k$ evolves through the *uncertain* dynamics (3.9) starting from the origin. It is easy to verify that dynamics (3.5) is the superposition of dynamics (3.6), (3.8) and (3.9).

3.2.3 Tube Parameterization

The effect of the uncertainty $\tilde{s}_k$ is accounted for by employing the set-dynamics

$$\forall k \in \mathbb{N}_{N-1}, \quad \widetilde{S}_{k+1} = A_K\widetilde{S}_k \oplus \mathcal{W} \quad \text{with} \quad \widetilde{S}_0 = \{0\}, \tag{3.10}$$

induced by the local uncertainty dynamics (3.9) and the disturbance set $\mathcal{W}$. It follows from (3.10) that

$$\forall k \in \mathbb{N}_{[1,N]}, \quad \widetilde{S}_k = \bigoplus_{i=0}^{k-1} A_K^i\mathcal{W} \quad \text{and} \quad \widetilde{S}_0 = \{0\}. \tag{3.11}$$

The state tube is a sequence of sets, denoted by $\mathbf{X}_N := \{X_k\}_{k\in\mathbb{N}_N}$, in which each set $X_k \subseteq \mathbb{R}^n$ is parameterized via $z_k \in \mathbb{R}^n$, $\bar{s}_k \in \mathbb{R}^n$ and the set $\widetilde{S}_k \subseteq \mathbb{R}^n$ as follows

$$\forall k \in \mathbb{N}_N, \quad X_k := z_k + \bar{s}_k \oplus \widetilde{S}_k. \tag{3.12}$$

For all $k \in \mathbb{N}_N$, $z_k + \bar{s}_k$ and $\widetilde{S}_k$ are, respectively, the centers and cross-sections of the tube $\mathbf{X}_N$.

3.2.4 Chapter Objectives

As discussed in the critique article [31], exploring the differences of the competing methods [11], [12] is necessary to gain some inspirations for the current research on both robust and stochastic MPC. In this chapter, we introduce an improved tube MPC that combines best principal components of [11] and [12], as well as the rigid tube MPC [13]. More specifically, the improved tube MPC holds a prediction tube $\mathbf{X}_N$ with time-varying cross-sections identical with [11], while it makes use of the stage cost and terminal ingredients design from [12], [13]. The initialization of the nominal state prediction is compatible with [13].

3.3 Tube Constraints, Decision Variables and Cost Function

This section outlines a tube initialization strategy that utilizes the maximal robust positively invariant set. This section also discusses the tube stage and terminal constraints and their constructions with the aid of the support functions, and it specifies the set of admissible decision variables and the employed cost function.

3.3.1 Initialization

This chapter adopts the initialization strategy as follows

$$\bar{s}_0 = x - z_0 \quad \text{and} \quad \bar{s}_0 \in \mathcal{X}_f. \tag{3.13}$$

Assumption 3.3 *The set $\mathcal{X}_f \subseteq \mathbb{R}^n$ is the maximal robust positively invariant set for the system $x^+ = A_K x + w$, and constraints $(x, Kx) \in \mathcal{Y}$ and $w \in \mathcal{W}$.*

The maximal robust positively invariant set $\mathcal{X}_f$ is the limit of the following standard set iteration [32]

$$\begin{aligned} \forall j \in \mathbb{N}, \quad \mathcal{X}_{j+1} &:= \{x \ : \ A_K x \in \mathcal{X}_j \ominus \mathcal{W}\} \bigcap \mathcal{X}_0 \quad \text{with} \\ \mathcal{X}_0 &:= \{x \ : \ (x, Kx) \in \mathcal{Y}\}, \end{aligned}$$

i.e., $\mathcal{X}_f := \mathcal{X}_\infty$. It is known that the maximal robust positively invariant set [32] is nonempty if and only if the minimal robust positively invariant set $\mathcal{X}_\mathcal{O}$ for $x^+ = A_K x + w$

with $w \in \mathcal{W}$, specified by

$$\mathcal{X}_{\mathcal{O}} := \bigoplus_{j=0}^{\infty} A_K^j \mathcal{W}, \tag{3.14}$$

is a subset of $\mathcal{X}_0$. The polyhedral structure of the limit set $\mathcal{X}_\infty$, however, is not necessarily guaranteed when the limit is not finitely determined. It is also well known that the set $\mathcal{X}_\infty$ is finitely determined when one of the iterates $\mathcal{X}_j$ is bounded and $\mathcal{X}_{\mathcal{O}} \subseteq \operatorname{interior}(\mathcal{X}_0)$ [32]. More generally, the maximal robust positively invariant set is finitely determined when $\mathcal{X}_j \subseteq \mathcal{X}_{j+1}$ for some $j \in \mathbb{N}$, in which case $\mathcal{X}_\infty = \mathcal{X}_j$ is finitely determined and it is nonempty when, in addition, $\mathcal{X}_{\mathcal{O}} \subseteq \mathcal{X}_0$.

Throughout this chapter, we assume that $\mathcal{X}_f$ is finitely determined and nonempty (thus, $\mathcal{X}_f$ is at least a polyhedral proper D–set in $\mathbb{R}^n$) and $\mathcal{X}_{\mathcal{O}}$ is strictly admissible with respect to the stage constraints (i.e., that, for all $x \in \mathcal{X}_{\mathcal{O}}$, we have $(x, Kx) \in \operatorname{interior}(\mathcal{Y})$). These natural conditions are summarized by the following assumption.

Assumption 3.4

- *The set $\mathcal{X}_f \subseteq \mathbb{R}^n$ is a polyhedral proper D–set, and its irreducible representation is given, for known vectors $r_i \in \mathbb{R}^n,\ i \in \mathcal{I}_{\mathcal{X}_f} := \{1, 2, \ldots, n_f\}$ with $n_f \in \mathbb{N}_+$, by*

$$\mathcal{X}_f := \{x \ : \ \forall i \in \mathcal{I}_{\mathcal{X}_f},\ r_i^\top x \le 1\}. \tag{3.15}$$

- *The set $\mathcal{X}_{\mathcal{O}}$ is such that $\mathcal{X}_{\mathcal{O}} \subseteq \operatorname{interior}(\mathcal{X}_0)$, or, equivalently its support function satisfies*

$$\forall i \in \mathcal{I}_{\mathcal{Y}}, \quad \mathrm{h}\left(\mathcal{X}_{\mathcal{O}}, c_i + K^\top d_i\right) < 1. \tag{3.16}$$

3.3.2 Stage Constraints

Based on the tube cross-section sets X_k in (3.12) and the affine control laws $\pi_k(\cdot,\cdot,\cdot,\cdot)$ in (3.4), the tube stage constraints take the form

$$\begin{aligned} &\forall k \in \mathbb{N}_{N-1},\ (z_k, \bar{s}_k, v_k) \in \mathcal{G}_k \text{ with} \\ &\mathcal{G}_k := \left\{(z, \bar{s}, v) : \forall \tilde{s} \in \widetilde{S}_k,\ (z + \bar{s} + \tilde{s}, v + K\bar{s} + K\tilde{s}) \in \mathcal{Y}\right\}. \end{aligned} \tag{3.17}$$

To handle these constraints in a computationally practicable manner, we make use of the support functions of sets $\widetilde{S}_k$, which allows us to avoid the explicit implementation of the underlying set-algebraic operations. By applying [14, Proposition 1], the structural properties of sets $\mathcal{G}_k,\ k \in \mathbb{N}_{N-1}$ can be summarized by the following proposition.

Proposition 3.1 *Suppose Assumptions 3.1, 3.2 and 3.4 hold. For all $k \in \mathbb{N}_{N-1}$, the stage constraint sets $\mathcal{G}_k$ are polyhedral proper D–sets in $\mathbb{R}^{2n+m}$ with (possibly redundant) representations*

$$\mathcal{G}_k := \left\{(z, \bar{s}, v) : \forall i \in \mathcal{I}_{\mathcal{Y}},\, c_i^\top z + \eta_i^\top \bar{s} + d_i^\top v \leq 1 - f_{(k,i)}\right\},$$

where, for all $i \in \mathcal{I}_{\mathcal{Y}}$, with $\eta_i := c_i + K^\top d_i$, the scalars $f_{(k,i)} \in [0, 1)$ are specified by

$$f_{(k,i)} := \mathrm{h}\left(\widetilde{S}_k, \eta_i\right) = \sum_{j=0}^{k-1} \mathrm{h}\left(\mathcal{W},\, (A_K^j)^\top \eta_i\right). \tag{3.18}$$

Proof: The claims can be verified by directly applying [14, Proposition 1]. Specifically, Proposition 1 of [14] applies by replacing the underlying support functions therein with the evaluation of support functions $\mathrm{h}\left(\widetilde{S}_k, \eta_i\right)$ for all $k \in \mathbb{N}_N$ and all $i \in n_{\mathcal{Y}}$. In regard of $\widetilde{S}_k$ specified in (3.11), the equations (3.18) follows from the properties of support funtions (3.21)–(3.22), as also shown in Lemma 2 of [14]. □

3.3.3 Terminal Constraints

The tube terminal constraint takes the form

$$(z_N, \bar{s}_N) \in \mathcal{H}_f \quad \text{with} \quad \mathcal{H}_f := \left\{(z, \bar{s}) : \forall \tilde{s} \in \widetilde{S}_N,\; z + \bar{s} + \tilde{s} \in \mathcal{X}_f\right\}. \tag{3.19}$$

Analogously to the construction of sets $\mathcal{G}_k$ in Proposition 3.1, the structural properties of the set $\mathcal{H}_f$ can be summarized as per the following.

Proposition 3.2 *Suppose Assumptions 3.1–3.4 hold. The set $\mathcal{H}_f$ is a polyhedral proper D–set in $\mathbb{R}^{2n}$, with a (possibly redundant) representation*

$$\mathcal{H}_f := \{(z, \bar{s}) \,:\, \forall i \in \mathcal{I}_{\mathcal{X}_f},\, r_i^\top z + r_i^\top \bar{s} \leq 1 - g_i\},$$

where, for all $i \in \mathcal{I}_{\mathcal{X}_f}$, the scalars $g_i \in [0, 1)$ are specified by

$$g_i := \mathrm{h}\left(\widetilde{S}_N, r_i\right) = \sum_{j=0}^{N-1} \mathrm{h}\left(\mathcal{W},\, (A_K^j)^\top r_i\right). \tag{3.20}$$

Proof: The proof is analogous to that of Proposition 3.1. □

In the following two remarks, we discuss the computational aspects of the relevant scalars $f_{(k,i)}$ and g_i.

Remark 3.1 *As shown in* (3.18) *and* (3.20)*, the evaluation of the underlying scalars* $f_{(k,i)}$ *and* g_i *requires merely the evaluation of a number of support functions of* $\mathcal{W}$*. Moreover, these scalars can be computed by the following simple iterations given, with* $f_{(0,i)} = 0$ *and* $g_{(0,i)} = 0$*, by*

$$\begin{aligned} &\forall k \in \mathbb{N}_{N-2}, \quad \forall i \in \mathcal{I}_{\mathcal{Y}}, \quad f_{(k+1,i)} = f_{(k,i)} + \mathrm{h}\left(\mathcal{W}, (A_K^k)^\top \eta_i\right) \\ &\forall j \in \mathbb{N}_{N-1}, \quad \forall i \in \mathcal{I}_{\mathcal{X}_f}, \quad g_{(j+1,i)} = g_{(j,i)} + \mathrm{h}\left(\mathcal{W}, (A_K^j)^\top r_i\right), \end{aligned}$$

in which we set $g_i := g_{(N,i)}$*. This process needs one to evaluate* $\mathrm{h}\,(\mathcal{W}, \cdot)$ *for* $(N-1)n_{\mathcal{Y}}$ *times to calculate* $f_{(k,i)}$*,* $k \in \mathbb{N}_{[1:N-1]}$*,* $i \in \mathcal{I}_{\mathcal{Y}}$ *and to evaluate* $\mathrm{h}\,(\mathcal{W}, \cdot)$ *for* Nn_f *times to calculate* g_i*,* $i \in \mathcal{I}_{\mathcal{X}_f}$*.*

Remark 3.2 *For a polyhedral set given by*

$$\mathcal{X} = \{x \in \mathbb{R}^n : \forall i \in \mathcal{I},\ a_i^\top x \leq 1\},$$

where $a_i \in \mathbb{R}^n$ *for each* $i \in \mathcal{I}$ *and* $\mathcal{I}$ *is a finite index set, the associated support function is specified by*

$$\forall y \in \mathbb{R}^n, \quad \mathrm{h}\,(\mathcal{X}, y) = \sup_x \{y^\top x : \forall i \in \mathcal{I},\ a_i^\top x \leq 1\},$$

which is a simple linear programming problem. Likewise, for a polytopic set given by

$$\mathcal{X} = \mathrm{conv}\{\lambda_i \in \mathbb{R}^n : i \in \mathcal{I}\},$$

where $\mathcal{I}$ *is a finite index set and* conv *denotes the convex hull, the associated support function is specified by*

$$\forall y \in \mathbb{R}^n \quad \mathrm{h}\,(\mathcal{X}, y) = \max_{i \in \mathcal{I}} y^\top \lambda_i.$$

For a closed unit p*-norm ball* $\mathcal{B}_p^n$*, the associated support function is specified by*

$$\forall y \in \mathbb{R}^n \quad \mathrm{h}\left(\mathcal{B}_p^n, y\right) = \|y\|_q.$$

where $p \in (1,\infty)$ and $q \in (1,\infty)$ satisfies $\frac{1}{p}+\frac{1}{q}=1$, $q=\infty$ if $p=1$ and $q=1$ if $p=\infty$. For an ellipsoidal set

$$\mathcal{X} = \{x \in \mathbb{R}^n \, : \, (x-c)^\top M^{-1}(x-c) \leq r^2\}$$

with shape matrix $M = M^\top \succeq 0$, center $c \in \mathbb{R}^n$ and radius $r > 0$, the associated support function is specified by

$$\forall y \in \mathbb{R}^n, \quad \mathrm{h}\,(\mathcal{X}, y) = y^\top c + r\sqrt{y^\top M y}.$$

Additionally, for any nonempty closed convex sets $\mathcal{X}$, $\mathcal{X}_1$ and $\mathcal{X}_2$ in $\mathbb{R}^n$, and any matrix $M \in \mathbb{R}^{m\times n}$, it holds that

$$\forall y \in \mathbb{R}^n, \quad \mathrm{h}\,(\mathcal{X}_1 \oplus \mathcal{X}_2, x) = \mathrm{h}\,(\mathcal{X}_1, x) + \mathrm{h}\,(\mathcal{X}_2, y) \quad \textit{and} \tag{3.21}$$

$$\forall \bar{y} \in \mathbb{R}^m, \quad \mathrm{h}\,(M\mathcal{X}, \bar{y}) = \mathrm{h}\left(\mathcal{X}, M^\top \bar{y}\right). \tag{3.22}$$

3.3.4 Set of Admissible Decision Variables

Within our setting, for a state $x \in \mathbb{R}^n$, the variables

$$\mathbf{z}_N := (z_0, \ldots, z_N), \quad \bar{\mathbf{s}}_N := (\bar{s}_0, \ldots, \bar{s}_N) \quad \text{and} \quad \mathbf{v}_{N-1} := (v_0, \ldots, v_{N-1})$$

determine entirely the state tubes $\mathbf{X}_N$ and the related control policy $\Pi_{N-1}(\cdot)$, and, thus, form the decision variable

$$\mathbf{d}_N := (\mathbf{z}_N, \bar{\mathbf{s}}_N, \mathbf{v}_{N-1}) \in \mathbb{R}^{n_{\mathbf{d}_N}} \tag{3.23}$$

with $n_{\mathbf{d}_N} = 2(N+1)n + Nm$. The decision variable $\mathbf{d}_N$ is required to satisfy dynamical consistency constraints (3.6) and (3.8), tube initialization constraints (3.13), tube stage constraints (3.17) and tube terminal constraints (3.19), which are summarized as follows

$$\forall k \in \mathbb{N}_{N-1},\ z_{k+1} = Az_k + Bv_k, \tag{3.24a}$$

$$\forall k \in \mathbb{N}_{N-1},\ \bar{s}_{k+1} = A_K \bar{s}_k \text{ with } \bar{s}_0 = x - z_0, \tag{3.24b}$$

$$\forall i \in \mathcal{I}_{\mathcal{X}_f},\ r_i^\top \bar{s}_0 \leq 1, \tag{3.24c}$$

$$\forall k \in \mathbb{N}_{N-1},\ \forall i \in \mathcal{I}_{\mathcal{Y}},\ c_i^\top z_k + d_i^\top v_k + \eta_i^\top \bar{s}_k \leq 1 - f_{(k,i)} \text{ and} \tag{3.24d}$$

$$\forall i \in \mathcal{I}_{\mathcal{X}_f},\ r_i^\top z_N + r_i^\top \bar{s}_N \leq 1 - g_i. \tag{3.24e}$$

The constraints in (3.24) amount to $(2N+1)n$ affine equality constraints and $2n_f + Nn_{\mathcal{Y}}$ affine inequality constraints. The set of admissible decision variables $\mathcal{D}_N(x)$ is given, for a state $x \in \mathbb{R}^n$, by

$$\mathcal{D}_N(x) := \{\mathbf{d}_N \, : \, \text{relations (3.24) hold}\}. \tag{3.25}$$

3.3.5 Cost

The utilized overall cost $V_N(\cdot) : \mathbb{R}^{n_{\mathbf{d}_N}} \to \mathbb{R}_{\geq 0}$ associated with the tube $\mathbf{X}_N$ is specified, for all $\mathbf{d}_N \in \mathbb{R}^{n_{\mathbf{d}_N}}$, by

$$V_N(\mathbf{d}_N) := \sum_{k=0}^{N-1} \left(z_k^\top Q z_k + v_k^\top R v_k \right) + z_N^\top P z_N. \tag{3.26}$$

Assumption 3.5 *The matrices $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$ and $P \in \mathbb{R}^{n\times n}$ are all known, symmetric and positive definite, i.e., $Q = Q^\top \succ 0$, $R = R^\top \succ 0$ and $P = P^\top \succ 0$, and satisfy the following Lyapunov inequality*

$$A_K^\top P A_K \preceq P - \left(Q + K^\top R K \right). \tag{3.27}$$

Within the considered setting, there exist scalars $\beta_1 \in (0,\infty)$ and $\beta_3 \in (0,\infty)$ such that, for all $\mathbf{d}_N \in \mathbb{R}^{n_{\mathbf{d}_N}}$, the overall cost $V_N(\mathbf{d}_N)$ satisfies

$$\begin{aligned} &\beta_1 \|z_0\|^2 \leq z_0^\top Q z_0, \quad \beta_1 \|z_0\|^2 \leq \beta_1 \|(\mathbf{z}_N, \mathbf{v}_{N-1})\|^2 \quad \text{and} \\ &\beta_1 \|(\mathbf{z}_N, \mathbf{v}_{N-1})\|^2 \leq V_N(\mathbf{d}_N) \leq \beta_3 \|(\mathbf{z}_N, \mathbf{v}_{N-1})\|^2. \end{aligned} \tag{3.28}$$

3.4 Improved Tube Model Predictive Control

In this section, the tube optimal control problem is formulated and analyzed. Then, we present the tube model predictive controller and its control-theoretic properties.

3.4.1 Tube Optimal Control

For a current state $x \in \mathbb{R}^n$, the tube optimal control problem, labeled by $\mathfrak{P}_N(x)$, reduces to selection of $\mathbf{d}_N \in \mathcal{D}_N(x)$ that minimizes $V_N(\mathbf{d}_N)$ so that, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} V_N^0(x) &:= \min_{\mathbf{d}_N} \{ V_N(\mathbf{d}_N) \ : \ \mathbf{d}_N \in \mathcal{D}_N(x) \} \quad \text{and} \\ \mathbf{d}_N^0(x) &:= \arg\min_{\mathbf{d}_N} \{ V_N(\mathbf{d}_N) \ : \ \mathbf{d}_N \in \mathcal{D}_N(x) \}. \end{aligned}$$

Here, $\mathbf{d}_N$ is defined in (3.23), $V_N(\mathbf{d}_N)$ is defined in (3.26), and $\mathcal{D}_N(x)$ is specified by (3.25). The effective domain $\mathcal{C}_N$ of the value function $V_N^0(\cdot)$ and its optimizer $\mathbf{d}_N^0(\cdot)$, also known as the N–step controllable set, is

$$\mathcal{C}_N := \{ x \ : \ \mathcal{D}_N(x) \neq \emptyset \}.$$

The set of admissible decision variables $\mathcal{D}_N(x)$, for each $x \in \mathcal{C}_N$, is a nonempty, closed polyhedral subset of $\mathbb{R}^{n_{\mathbf{d}_N}}$. Thus, for any given $x \in \mathbb{R}^n$, $\mathfrak{P}_N(x)$ is a convex quadratic programming problem, which is feasible for all $x \in \mathcal{C}_N$. If we eliminate the dynamic consistency constraints (3.24b) and replace $\bar{s}_k$ with $A_K^k(x - z_0)$, for all $k \in \mathbb{N}_N$, the decision variable in (3.23) reduces to $\mathbf{d}_N = (\mathbf{z}_N, \mathbf{v}_{N-1})$. The cost function $(\mathbf{z}_N, \mathbf{v}_{N-1}) \to V_N(\mathbf{z}_N, \mathbf{v}_{N-1})$ is strictly convex and quadratic. Thus, for all $x \in \mathcal{C}_N$, $\mathfrak{P}_N(x)$ reduces to a strictly convex quadratic programming problem. In this context, the key properties of the value function $V_N^0(\cdot)$, the optimizer maps $\mathbf{z}_N^0(\cdot)$ and $\mathbf{v}_{N-1}(\cdot)$, and their effective domain $\mathcal{C}_N$ can be summarized by the following theorem.

Theorem 3.1 *Suppose Assumptions 3.1–3.5 hold, and take any $N \in \mathbb{N}$.*

- *The value function $V_N^0(\cdot)$ is continuous, convex, piecewise quadratic and such that:*

$$\forall x \in \mathcal{X}_f, \quad V_N^0(x) = 0; \tag{3.29}$$

$$\forall x \in \mathcal{C}_N \setminus \mathcal{X}_f, \quad 0 < V_N^0(x) < \infty. \tag{3.30}$$

- *The optimizer maps $\mathbf{z}_N^0(\cdot)$ and $\mathbf{v}_{N-1}^0(\cdot)$ are continuous, piecewise affine and such that:*

$$\forall x \in \mathcal{X}_f, \quad \left\|\left(\mathbf{z}_N^0(x), \mathbf{v}_{N-1}^0(x)\right)\right\| = 0 \quad \textit{and} \quad \|z_0^0(x)\| = 0, \tag{3.31}$$

$$\forall x \in \mathcal{C}_N \setminus \mathcal{X}_f, \quad \left\|\left(\mathbf{z}_N^0(x), \mathbf{v}_{N-1}^0(x)\right)\right\| > 0 \quad \textit{and} \quad \|z_0^0(x)\| > 0. \tag{3.32}$$

- *The effective domain $\mathcal{C}_N \subseteq \mathbb{R}^n$ is a polyhedral proper D–set and satisfies*

$$\mathcal{X}_f = \mathcal{C}_0 \subseteq \ldots \subseteq \mathcal{C}_N \subseteq \mathcal{Y}_x, \tag{3.33}$$

where $\mathcal{Y}_x$ is the $(x, u) \mapsto x$ projection of the set $\mathcal{Y}$.

Proof: The related topological statements follow from the standard properties of the solution to the considered parametric convex quadratic programming problem $\mathfrak{P}_N(x)$ and its construction, see [2], [13], [33] for more details. Since $x \in \mathcal{X}_f$, it follows from choosing $z_0(x) = x - \bar{s}_0(x) = 0$ with $\bar{s}_0(x) = x \in \mathcal{X}_f$ that $z_N(x) = \ldots = z_0(x) = 0$ and $v_{N-1}(x) = \ldots = v_0(x) = 0$ are feasible for $\mathfrak{P}_N(x)$. Hence $V_N^0(x) \leq V_N(\mathbf{d}_N(x)) = 0$, with $\mathbf{d}_N(x) = (0, \ldots, 0, 0, \ldots, 0, x, A_K x, \ldots, A_K^{N-1} x)$, which ensures that (3.29) and (3.31) hold. Likewise, if $x \in \mathcal{C}_N \setminus \mathcal{X}_f$, the constraint $\bar{s}_0(x) \in \mathcal{X}_f$ implies that $z_0(x) = x - \bar{s}_0(x) \neq 0$ such that (3.30) and (3.32) hold. The inclusion relation (3.33) holds true by applying [2, Proposition 2.10]. $\square$

3.4.2 Tube MPC

The tube model predictive control evaluates implicitly the control law $\kappa_N(\cdot)$, specified by

$$\forall x \in \mathcal{C}_N, \quad \kappa_N(x) := v_0^0(x) + K\left(x - z_0^0(x)\right), \tag{3.34}$$

and it induces the tube model predictive controlled uncertain dynamics

$$\forall x \in \mathcal{C}_N, \quad x^+ \in Ax + B\kappa_N(x) \oplus \mathcal{W}. \tag{3.35}$$

In view of (3.31) in Theorem 3.1, for any $x \in \mathcal{X}_f$, $z_0^0(x) = 0$ and $v_0^0(x) = 0$. Thus, for all states x in the terminal constraint set $\mathcal{X}_f$, the tube model predictive control law $\kappa_N(\cdot)$ and its controlled uncertain dynamics (3.35) satisfy

$$\forall x \in \mathcal{X}_f, \quad \kappa_N(x) = Kx \quad \text{and} \quad x^+ \in A_K x \oplus \mathcal{W}. \tag{3.36}$$

3.4.3 Robust Positive Invariance and Stability

In our setting, Theorem 3.2 establishes the robust positive invariance property of the effective domain $\mathcal{C}_N$ (also known as the robust recursive feasibility of the tube optimal contol problem $\mathfrak{P}_N(\cdot)$).

Theorem 3.2 *Suppose Assumptions 3.1–3.5 hold, and take any $N \in \mathbb{N}$. The effective domain $\mathcal{C}_N$ is a robust positively invariant set for the dynamics* (3.35) *and implicitly induced constraints* $(x, \kappa_N(x)) \in \mathcal{Y}$.

Proof: When $N = 0$, the statement holds directly by the robust positively invariance of the set $\mathcal{X}_f$ and the controlled dynamics (3.36). For any $N \in \mathbb{N}_+$, suppose $x \in \mathcal{C}_N$ so that $\mathfrak{P}_N(x)$ is feasible and the optimizer $\mathbf{d}_N^0(x) = \left(\mathbf{z}_N^0(x), \bar{\mathbf{s}}_N^0(x), \mathbf{v}_{N-1}^0(x)\right)$ exists. For all $w \in \mathcal{W}$, it holds that

$$x^+ = Ax + B\kappa_N(x) + w = z_1^0(x) + \bar{s}_1^0(x) + w.$$

Let us define

$$\mathbf{d}_N(x^+) := \left(\mathbf{z}_N(x^+), \bar{\mathbf{s}}_N(x^+), \mathbf{v}_{N-1}(x^+)\right), \tag{3.37}$$

in which

$$\begin{aligned}
&\mathbf{z}_N(x^+) = (z_1^0(x), \ldots, z_N^0(x), A_K z_N^0(x)),\\
&\mathbf{s}_N(x^+) = (\bar{s}_1^0(x) + w, ..., \bar{s}_N^0(x) + A_K^{N-1} w, A_K \bar{s}_N^0(x) + A_K^N w),\\
&\mathbf{v}_{N-1}(x^+) = (v_1^0(x), \ldots, v_{N-1}^0(x), K z_N^0(x)).
\end{aligned}$$

In the sequel, we will prove that $\mathbf{d}_N(x^+)$ is feasible for $\mathfrak{P}_N(x^+)$, i.e., $\mathbf{d}_N(x^+) \in \mathcal{D}_N(x^+)$. The dynamic consistency constraints (3.24a)–(3.24b) are satisfied by the construction of $\mathbf{d}_N(x^+)$. The tube initialization constraint $\bar{s}_0(x^+) = \bar{s}_1^0(x) + w = A_K\bar{s}_0^0(x) + w \in \mathcal{X}_f$ is satisfied by $\bar{s}_0^0(x) \in \mathcal{X}_f$ and the robust invariance property of $\mathcal{X}_f$, i.e., the inequality constraints (3.24c) is satisfied. For all $k \in \mathbb{N}_{N-1}$, all $w \in \mathcal{W}$ and all

$$
\begin{aligned}
&\forall k \in \mathbb{N}_{N-1},\ \forall w \in \mathcal{W},\ \forall \tilde{s} \in \widetilde{S}_k: \\
&\qquad z_k(x^+) + \bar{s}_k(x^+) + \tilde{s} = z_{k+1}^0(x) + \bar{s}_{k+1}^0(x) + A_K^k w + \tilde{s} \\
&\qquad v_k(x^+) + K\bar{s}_k(x^+) + K\tilde{s} = v_{k+1}^0(x) + K\bar{s}_{k+1}^0(x) + KA_K^k w + K\tilde{s}
\end{aligned}
$$

It follows from (3.10) that for all $w \in \mathcal{W}$ and all $\tilde{s} \in \widetilde{S}_k$, $A_K^k w + \tilde{s} \in \widetilde{S}_{k+1}$. Therefore, it further follows that

$$
\forall \hat{s} \in \widetilde{S}_{k+1},\ (z_{k+1}^0(x) + \bar{s}_{k+1}^0(x) + \hat{s}, v_{k+1}^0(x) + K\bar{s}_{k+1}^0(x) + K\hat{s}) \in \mathcal{Y}
$$

by the definitions of stage constraints (3.17) and terminal constraints (3.19). Thus, we have that

$$
\forall k \in \mathbb{N}_{N-1}, \quad \forall \tilde{s} \in \widetilde{S}_k,\ (z_k(x^+) + \bar{s}_k(x^+) + \tilde{s}, v_k(x^+) + K\bar{s}_k(x^+) + K\tilde{s}) \in \mathcal{Y},
$$

i.e., the inequality constraints (3.24d) are satisfied. For $k = N$,

$$
\begin{aligned}
z_N(x^+) + \bar{s}_N(x^+) \oplus \widetilde{S}_N &= A_K z_N^0(x) + A_K\bar{s}_N^0(x) + A_K^N w \oplus \widetilde{S}_N \\
&\subseteq A_K z_N^0(x) + A_K\bar{s}_N^0(x) \oplus \widetilde{S}_{N+1}
\end{aligned}
$$

Since $z_N^0(x) + \bar{s}_N^0(x) \oplus \widetilde{S}_N \subseteq \mathcal{X}_f$ and by the robust invariance property of $\mathcal{X}_f$,

$$
A_K(z_N^0(x) + \bar{s}_N^0(x) \oplus \widetilde{S}_N) \oplus \mathcal{W} = A_K z_N^0(x) + A_K\bar{s}_N^0(x) \oplus \widetilde{S}_{N+1} \subseteq \mathcal{X}_f.
$$

Thus, the tube terminal constraints $z_N(x^+) + \bar{s}_N(x^+) \oplus \widetilde{S}_N \subseteq \mathcal{X}_f$ are satisfied, i.e., the inequality constraints (3.24e) are satisfied. Hence, we have that $\mathbf{d}_N(x^+) \in \mathcal{D}_N(x^+)$, which further implies that $\forall w \in \mathcal{W}$, $x^+ = Ax + B\kappa_N(x) + w \in \mathcal{C}_N$ and completes the proof. $\square$

In our setting, the lower and upper bounds of the overall cost $V_N(\cdot)$ in (3.28) and Theorem 3.1 yield the following.

Proposition 3.3 *Suppose Assumptions 3.1–3.5 hold, and take any $N \in \mathbb{N}$. There exist two scalars $\beta_1 \in (0, \infty)$ and $\beta_2 \in (0, \infty)$ such that, for all $x \in \mathcal{C}_N$,*

$$
\beta_1\|z_0^0(x)\|^2 \le V_N^0(x) \le \beta_2\|z_0^0(x)\|^2, \tag{3.38}
$$

and, for all $x \in \mathcal{C}_N$ and all $x^+ \in Ax + B\kappa_N(x) \oplus \mathcal{W}$,

$$
V_N^0(x^+) \le V_N^0(x) - \beta_1\|z_0^0(x)\|^2. \tag{3.39}
$$

Proof: The admissible decision variable $\mathbf{d}_N(x^+)$ specified in (3.37) yields the desired cost decrease, i.e.,

$$\begin{aligned}\forall x \in \mathcal{C}_N, \quad V_N^0(x^+) - V_N^0(x) &\leq V_N(x^+) - V_N^0(x) \\ &= \|z_N^0(x)\|_Q^2 + \|v_N^0(x)\|_R^2 + \|A_K z_N^0(x)\|_P^2 \\ &\quad - \|z_N^0(x)\|_P^2 - \|z_0^0(x)\|_Q^2 - \|v_0^0(x)\|_R^2 \\ &\leq -\|z_0^0(x)\|_Q^2 - \|v_0^0(x)\|_R^2,\end{aligned}$$

where the last inequality holds due to Assumption 3.5. Since there exists $\beta_1 > 0$ such that $\beta_1\|z_0^0(x)\|^2 \leq \|z_0^0(x)\|_Q^2 + \|v_0^0(x)\|_R^2$, it follows that

$$\forall x \in \mathcal{C}_N, \quad V_N^0(x^+) - V_N^0(x) \leq -\beta_1\|z_0^0(x)\|^2. \tag{3.40}$$

It follows from (3.28) that, for all $x \in \mathcal{C}_N$, there also exists $\beta_3 > 0$ such that

$$\beta_1\|z_0^0(x)\|^2 \leq V_N^0(x) \leq \beta_3\|(\mathbf{z}_N^0(x), \mathbf{v}_{N-1}^0(x))\|^2. \tag{3.41}$$

In light of Theorem 3.1, the optimizer maps $\mathbf{z}_N^0(\cdot)$ and $\mathbf{v}_{N-1}^0(\cdot)$ are Lipschitz continuous since they are continuous and piecewise affine functions defined on the nonempty closed polyhedral set $\mathcal{C}_N$. Thus, we have that for all $x \in \mathcal{C}_N$ and all $y \in \mathcal{C}_N$,

$$\|(\mathbf{z}_N^0(x), \mathbf{v}_{N-1}^0(x)) - (\mathbf{z}_N^0(y), \mathbf{v}_{N-1}^0(y))\| \leq L\|x - y\|, \tag{3.42}$$

in which $L > 0$ denotes the associated Lipschitz constant. When $x \in \mathcal{C}_N$ is the current state and $y := \bar{s}_0^0(x) \in \mathcal{X}_f \subseteq \mathcal{C}_N$, we have $(\mathbf{z}_N^0(y), \mathbf{v}_{N-1}^0(y)) = \mathbf{0}$ and $x - y = x - \bar{s}_0^0(x) = z_0^0(x)$ so that (3.42) yields the following

$$\forall x \in \mathcal{C}_N, \quad \|(\mathbf{z}_N^0(x), \mathbf{v}_{N-1}^0(x))\| \leq L\|z_0^0(x)\|. \tag{3.43}$$

It follows from (3.41) and (3.43) that, with $\beta_2 := \beta_3 L^2$,

$$\forall x \in \mathcal{C}_N, \quad \beta_1\|z_0^0(x)\|^2 \leq V_N^0(x) \leq \beta_2\|z_0^0(x)\|^2. \tag{3.44}$$

Thus, in view of (3.44) and (3.40), the proposition is proved. $\square$

The main, and relatively direct, ramification of the Proposition 3.3 and Theorem 3.1 and 3.2 is the following.

Theorem 3.3 *Assumptions 3.1–3.5 hold, and take any $N \in \mathbb{N}$. The maximal robust positively invariant set $\mathcal{X}_f$ is robustly exponentially stable for the dynamics* (3.35) *with the domain of attraction being equal to $\mathcal{C}_N$.*

Proof: In light of [2, Theorem B.19], Theorems 3.1–3.2 and Proposition 3.3 yield the statements of Theorem 3.3. □

When A_K satisfies Assumption 3.2, it is well-known that the minimal robust positively invariant set $\mathcal{X}_\mathcal{O}$ is robustly exponentially stable for the uncertain dynamics $x^+ \in A_K x \oplus \mathcal{W}$ with the domain of attraction being equal to the maximal robust positively invariant set $\mathcal{X}_f$. In light of (3.36), Theorem 3.3 and a local application of the analysis in [34], the following refinement of the robust exponential stability is affirmative.

Corollary 3.1 *Suppose Assumptions 3.1–3.5 hold, and take any $N \in \mathbb{N}$. The set $\mathcal{X}_\mathcal{O}$ is robustly exponentially stable for the dynamics* (3.35) *with the domain of attraction being equal to $\mathcal{C}_N$.*

Proof: Let the set sequence $\{\mathcal{Q}_k \subseteq \mathbb{R}^n\}_{k\in\mathbb{N}}$ be generated by the following set-dynamics

$$\forall k \in \mathbb{N},\ \mathcal{Q}_{k+1} = A_K \mathcal{Q}_k \oplus \mathcal{W} \text{ with } \mathcal{Q}_0 \subseteq \mathcal{X}_f.$$

Proposition 4.3 of [34] directly implies that, for any set $\mathcal{Q}_0$ in the set of compact subsets of $\mathcal{X}_f$, $\{\mathcal{Q}_k\}_{k\in\mathbb{N}}$ exponentially convergence to $\mathcal{X}_\mathcal{O}$ in a stable manner with respect to the Hausdorff distance. Using the facts of Theorem 3.3, the set $\mathcal{X}_\mathcal{O}$ is robustly exponentially stable for (3.35) with domain of attraction being equal to $\mathcal{C}_N$. □

3.5 Discussion

This section first provides a detailed discussion of the similarities and differences for the proposal in this chapter and the early proposals [11]–[13]. Then, we do numerical simulations on two case studies to highlight the benefits of the proposed method.

3.5.1 Comparison

State Decomposition and Control Policy: Unlike the proposal in this chapter decomposing the state predictions into three components, [11]–[13] decompose the state predictions into two components $\forall k \in \mathbb{N}_N,\ x_k = z_k + s_k$ with s_k denoting the local uncertainty. Our method and [12], [13] use the control parameterization (3.4), while the control parameterization of [11] is defined as

$$\forall k \in \mathbb{N}_{N-1},\ u_k = K x_k + \mu_k,$$

where $\mu_k \in \mathbb{R}^m$ is a control offset. However, the latter is recovered by a directly algebraic change of variables

$$\forall k \in \mathbb{N}_{N-1},\ \mu_k = v_k - Kz_k.$$

Cost Function: The proposal in this chapter and [12], [13] penalize the nominal state and control predictions, as defined in (3.26). Proposal [11] penalizes the control offsets, given by

$$J_N(\boldsymbol{\mu}_{N-1}) = \sum_{k=0}^{N-1} \mu_k^\top \Psi \mu_k = \sum_{k=0}^{N-1} (v_k - Kz_k)^\top \Psi (v_k - Kz_k),$$

where $\boldsymbol{\mu}_{N-1} := (\mu_0, \ldots, \mu_{N-1})$ and $\Psi = \Psi^\top \succ 0$. Within our setting, if K and P are such that (3.27) holds true with equality. Then it holds that

$$J_N(\boldsymbol{\mu}_{N-1}) = V_N(\mathbf{d}_N) - z_0^\top Q z_0 \quad \text{with} \quad \Psi = R + B^\top PB.$$

However, $J_N(\cdot)$ discards the cost term $z_0^\top Q z_0$ with $z_0 = x$ from $V_N(\cdot)$, which results in a value function $J_N^0(\cdot)$ that is not guaranteed to admit adequate upper bound necessary for establishing stronger robust exponential stability properties. Consequently, [11] only establishes robust convergence results with respect to $\mathcal{X}_\mathcal{O}$ albeit the authors in [35] proved that the tube MPC proposal of [11] is robust asymptotically stable.

Initialization: Proposal [11] enforces an initial condition $z_0 = x$. This is a disadvantage of [11] and, in fact, the main cause of the lack of robust stability guarantees. Proposal [12] enforces an initial condition $z_0 = z_1^\star$, where $z_1^\star$ is the predicted nominal state one-step ahead of the previous time instant. Proposal [13] allows for an initial condition $x - z_0 \in \mathcal{S}$, where $\mathcal{S}$ is an outer invariant approximation of the minimal robust positively invariant set for $s^+ = A_K s + w$, $w \in \mathcal{W}$. This initial condition facilitates a direct geometric argument for robust exponential stability of the set $\mathcal{S}$. In this regard, the improved proposal in this chapter refines all proposals [11]–[13] by allowing for a relaxed initial condition $x - z_0 \in \mathcal{X}_f$.

Tubes Stage Constraints: Proposals [12], [13] make uses of tubes with rigid cross-section $\mathcal{S}$, while the proposal in this chapter and [11] make uses of tubes with time-varying cross-sections $\widetilde{S}_k$, $k \in \mathbb{N}_N$. Tubes with time–varying cross-sections are less conservative since $\widetilde{S}_k \subseteq \mathcal{S}$ for all $k \in \mathbb{N}_N$. Thus, the utilization of time–varying cross-sections $\widetilde{S}_k$ reduces the effects of the uncertainty on the constraints. In order to modify stage constraints, for all $i \in n_\mathcal{Y}$, support function $\mathrm{h}\left(\widetilde{S}_k, \eta_i\right)$ needs to be evaluated offline for all $k \in \mathbb{N}_{N-1}$ for tubes with time-varying cross-sections $\widetilde{S}_k$, while for all $i \in n_\mathcal{Y}$, each support function $\mathrm{h}(\mathcal{S}, \eta_i)$ needs to be evaluated offline only once for rigid tubes with constant cross-sections $\mathcal{S}$. In this regard, a concrete comparison of the involved computation cost depends on the horizon N and the specific representations of sets $\widetilde{S}_k$ and $\mathcal{S}$.

Tubes Terminal Constraints: The terminal constraints in this chapter and [11] are specified by

$$z_N + \bar{s}_N \in \mathcal{X}_f \ominus \widetilde{S}_N \quad \text{and} \quad z_N \in \mathcal{X}_f \ominus \widetilde{S}_N,$$

respectively. Both proposals [12], [13] make use of terminal constraints $z_N \in \mathcal{Z}_f$, where $\mathcal{Z}_f$ is the maximal positively invariant set for system $z^+ = (A + BK_f)z$ and constraints $z \in \{z : \forall s \in \mathcal{S},\ (z+s, K_f(z+s)) \in \mathcal{Y}\}$. Note that in [12], [13], K_f is set to be the same as K, but it is possible to choose a matrix K that is different from K_f, as pointed out in [36]. When $K_f = K$, it is not difficult to verify that $\mathcal{Z}_f \oplus \mathcal{S} \subseteq \mathcal{X}_f$ since both sets $Z_f \oplus \mathcal{S}$ and $\mathcal{X}_f$ are robust positively invariant [36] and $\mathcal{X}_f$ is the maximal one. Since for all $N \in \mathbb{N}$, $\widetilde{S}_N \subseteq \mathcal{S}$, it holds that

$$\mathcal{Z}_f \subseteq \mathcal{X}_f \ominus \mathcal{S} \subseteq \mathcal{X}_f \ominus \widetilde{S}_N.$$

Thus, compared to [12], [13], terminal constraints used in this chapter and [11] bring about a further reduction of the effects of the uncertainty on the constraints.

Remark 3.3 *Tuning the control feedback K and the local terminal feedback K_f separately for [12], [13] provides more flexibility, and may results in a larger feasible domain. In [19], the authors have presented a scheme similar to [11] where K is not necessarily equal to K_f. However, although K_f given by the solution to the unconstrained infinite horizon linear quadratic regulator (A, B, Q, R) is optimal, a systematic way to optimally choose matrix K, in order to enlarge the effective domain and decrease the value function simultaneously, is still not yet available.*

Some key features of these proposals are summarized in Table 3.1. The robust stability and convergence is in terms of the minimal robust positively invariant set $\mathcal{X}_{\mathcal{O}}$. The effective domains of the four listed methods are compared in a relative sense. As already pointed out, proposal [11] has been proved to be robust asymptotically stable in [35]. The explicit construction of $\mathcal{Z}_f$ and $\mathcal{S}$ can be further avoided by making use of implicit set representations at the cost of introducing some extra optimization variables, as shown in [14]. The interested readers are refereed to [31] for a more detailed comparison between [11] and [12].

Table 3.1: Features of tube MPC proposals

method	offline computation	robust positive invariance	effective domain	robust stability and convergence
This Chapter	$\mathcal{X}_f$	yes	large	exponential stability
[11]	$\mathcal{X}_f$ and $\widetilde{S}_k$	yes	large	convergence
[12]	$\mathcal{Z}_f$ and $\mathcal{S}$	yes	small	asymptotic stability
[13]	$\mathcal{Z}_f$ and $\mathcal{S}$	yes	medium	exponential stability

Online Computation: The tube optimal control problems of [11]–[13] are strictly convex quadratic programming problems. For the proposal in this chapter, $\mathfrak{P}_N(x)$ reduces to a strictly convex quadratic programming problem with the use of the change of variables $\bar{s}_k = A_K^k(x - z_0)$ for all $k \in \mathbb{N}_N$. In terms of these strictly convex forms of $\mathfrak{P}_N(\cdot)$ and if the nominal dynamics in [12], [13] were not eliminated in order to keep the sparse structure of $\mathfrak{P}_N(\cdot)$, the number of decision variables and constraints are listed in Table 3.2. The notations $n_{\mathcal{Z}_f}$ and $n_{\mathcal{S}}$ denote the number of the inequality representations of the sets $\mathcal{Z}_f$ and $\mathcal{S}$, respectively. As shown in Table 3.2, the numbers of decision variables and constraints scale linearly with respect to horizon length N for all proposals, indicating that the online computations of all proposals are of the same order of complexity. As also shown in Table 3.2, the online computations are almost identical for all proposals, especially when N is large.

Table 3.2: Number of Decision Variables and Constraints

method	# of decision variables	# of affine equalities	# of affine inequalities
This chapter	$(N+1)n + Nm$	Nn	$2n_f + Nn_\mathcal{Y}$
[11]	Nm	$Nn + Nm$	$n_f + Nn_\mathcal{Y}$
[12]	$Nn + Nm$	Nn	$n_{\mathcal{Z}_f} + Nn_\mathcal{Y}$
[13]	$(N+1)n + Nm$	Nn	$n_\mathcal{S} + n_{\mathcal{Z}_f} + Nn_\mathcal{Y}$

3.5.2 Numerical Illustration

Example 3.1 *Our first example is taken from the constrained double integrator considered in [13], with the only exception that the stage constraint set is changed to*

$\mathcal{Y} := \mathbb{X} \times \mathbb{U}$ *and*

$$\mathbb{X} := \{x \in \mathbb{R}^2 \ : \ \|x\|_\infty \leq 100 \textit{ and } [0\ 1]x \leq 2\} \textit{ and}$$
$$\mathbb{U} := \{u \in \mathbb{R} \ : \ -1 \leq u \leq 1\}.$$

The terminal weighting matrix P and the feedback gain matrix K are obtained as the solutions to the infinite horizon unconstrained optimal control for (A, B, Q, R).

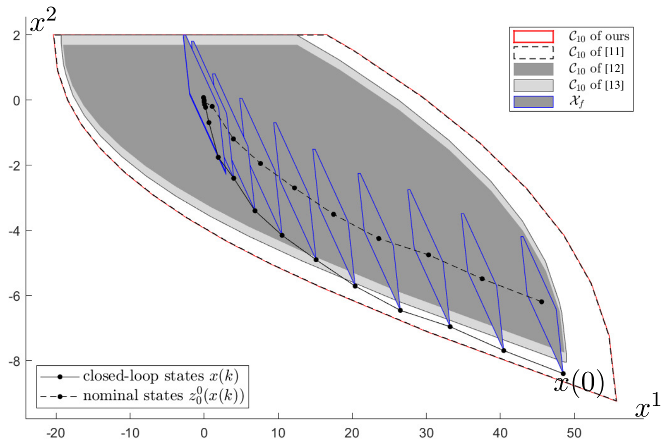

Figure 3.1: Effective domains and closed-loop simulations

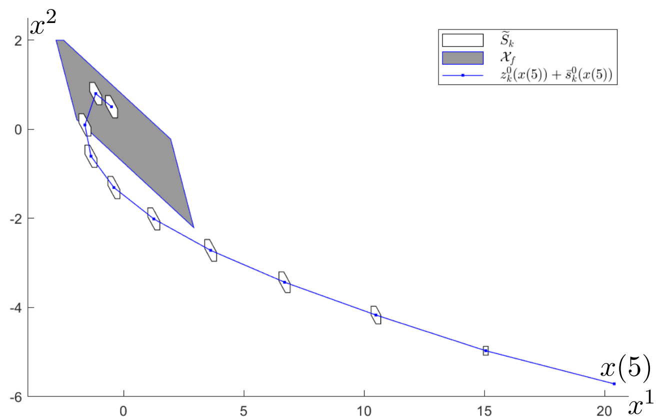

Figure 3.2: The Predicted tube $\mathbf{X}_N$ at the closed-loop time step 5.

We choose a prediction horizon $N = 10$ and start from an initial state $x(0) = [48.5; -8.4]$. In Figure 3.1, the effective domains of our method and [11] are identical, and they include the effective domains of [12], [13]. For 1000 sampled disturbance sequences, the closed-loop state $x(k)$ converges to the maximal robust positively invariant set $\mathcal{X}_f$ in 11 steps and to the minimal robust positively invariant set $\mathcal{X}_\mathcal{O}$ in 12 steps. Figure 3.2

visualizes the predicted tube $\mathbf{X}_{10}$ at sampling instant 5, and it shows that the required tube constraints are satisfied. We start from an initial state $x(0) = [48; -7]$ and simulate the control processes for [12], [13] and the proposal in this chapter with 1000 times. Figure 3.3 shows the the averaged value functions, labeled by $\bar{V}_{10}^{0}(\cdot)$, illustrating that our methods achieves lowest cost and faster convergences compared to [12], [13]. We remark that the cost function $J_N(\cdot)$ of [11] is defined on the control offsets $\boldsymbol{\mu}_{N-1}$, as discussed in section 3.5.1. Thus, we do not include the comparison of the corresponding results of [11] in Figure 3.3 for the sake of fairness.

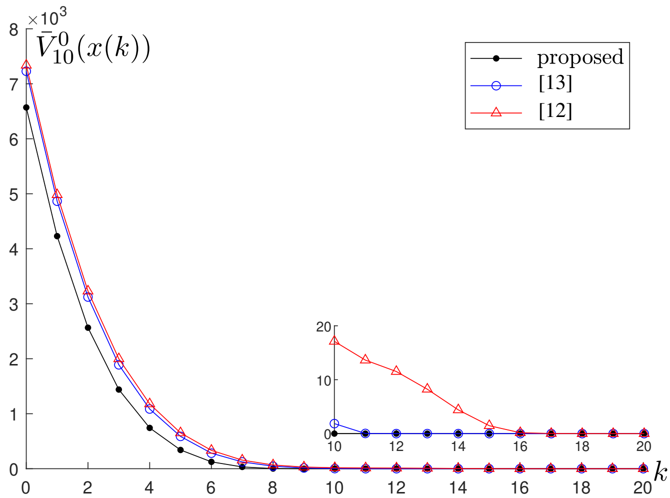

Figure 3.3: Averaged value functions $\bar{V}_{10}^{0}(\cdot)$ along $\{x(k)\}_{k\in\mathbb{N}_{20}}$.

To highlight the benefits of using support functions, we consider the following example in higher dimensions.

Example 3.2 *Our second example is taken from [14, Section 6.2], which is a variation of a modern transport airplane model in [37, Example AC9]. In this example,* $n = 10$ *and* $m = 4$*. The discrete time system matrices* A *and* B *are obtained via the Euler discretization with sampling period* $T = 0.5$ *[s]. The relevant sets* $\mathcal{Y}$ *and* $\mathcal{W}$*, and matrices* Q *and* R *are given by,* $\mathcal{Y} = 500\mathcal{B}_{\infty}^{10} \times 50\mathcal{B}_{\infty}^{4}$, $\mathcal{W} = \mathcal{B}_{2}^{10}$, $Q = 100I$ *and* $R = I$*. The terminal weighting matrix* P *and the feedback gain matrix* K *are also obtained as the solutions to the infinite horizon unconstrained optimal control for* (A, B, Q, R)*. The prediction horizon is specified by* $N = 20$*.*

For offline implementation in MATLAB, the maximal robust positively invariant set $\mathcal{X}_f$

is computed in 0.55 [s], and all the relevant scalars $f_{(k,i)}$ and g_i is computed in 0.008 [s] by using `NORM` function. Thus, the offline design is performed successfully in less than 0.56 seconds. For online implementation in MATLAB with `QUADPROG` solver, we start the simulation from an initial state $x(0) = [8; 50; 50; 100; 7; 15; 100; 2; 100; 50]$ for 20 time steps with a randomly sampled disturbances sequence. The average time to solve each of the considered quadratic programming problems is 0.08 [s]. Figure 3.4 depicts the simulated sequence $\{z_0^0(x(k))\}_{k\in\mathbb{N}_{19}}$ and the corresponding closed-loop state trajectory $\{x(k)\}_{k\in\mathbb{N}_{20}}$, illustrating that the proposed tube model predictive controlled states convergence exponentially fast to the associated minimal robust positively invariant set $\mathcal{X}_{\mathcal{O}}$.

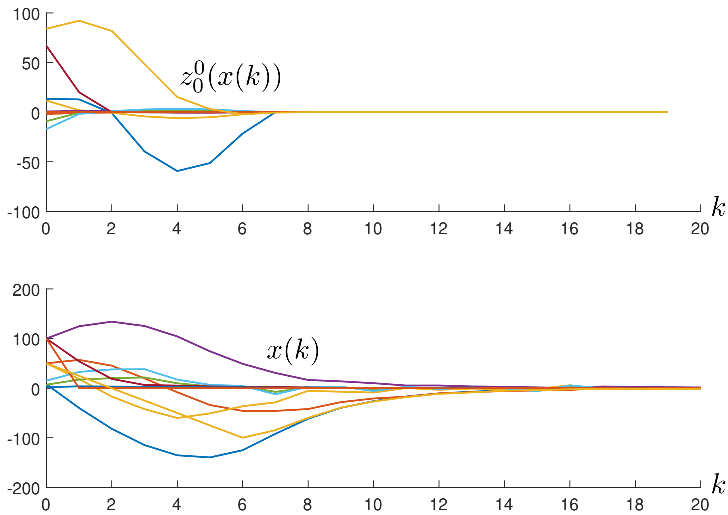

Figure 3.4: $\{z_0^0(x(k))\}_{k\in\mathbb{N}_{19}}$ and $\{x(k)\}_{k\in\mathbb{N}_{20}}$

3.6 Concluding Remarks

This chapter has revisited the early tube MPC methods [11]–[13] for more general mixed constraint set $\mathcal{Y}$ and disturbance set $\mathcal{W}$, and presented a refined tube MPC, which preserves all desirable computational and structural properties of its predecessors. The refined tube MPC also improves the desirable control-theoretic properties of [11]–[13] to a reasonable extent and simplifies the offline computations with the help of support functions. An important direction for future research involves extending the proposed refinement to incorporate a wider spectrum of control systems and adopting less conservative parameterized prediction schemes.

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Chapter 4

Robustifying MPC of Stage-Wise Chance Constrained Linear Systems

This Chapter is based on the following paper under review:

Kai Wang, Kiet Tuan Hoang and Sébastien Gros. Robustifying Model Predictive Control of Uncertain Linear Systems with Chance Constraints. *Accepted to IEEE Conference on Decision and Control (CDC)*, 2024.

4.1 Introduction

In the last two decades, model predictive control (MPC) has emerged as one of the most prevalent optimization-based control techniques in academia and industry [1]–[3]. The main advantages of MPC come from its ability to simultaneously cope with constraints, optimize performance and ensure desirable control-theoretic properties [4], [5]. However, many of these properties deteriorate in the presence of disturbances although, nominal (certainty-equivalent) MPC exhibits inherent robustness to some extent [6]. Nevertheless, an MPC that explicitly considers uncertainty is required in safety-critical applications. When the uncertainty description of the disturbance is available, robust MPC [7], [8], stochastic MPC [9], and more recent distributionally robust MPC [10] can be employed to maintain safety to certain acceptable levels. Whereas robust MPC considers the set-membership description of bounded uncertainties, stochastic MPC and distributionally robust MPC consider uncertainties which follow specific probability distributions or sets of probability distributions, respectively.

To alleviate the computational complexity of the exact min-max robust MPC [5], tube MPC has been widely investigated as a sensible option for its intuitive simplicity, computational practicality and guaranteed control-theoretic properties. Based on the two early tube MPC methods for linear systems under bounded disturbance [11], [12] which were proposed almost simultaneously, many subsequent studies have appeared for both tube-based robust MPC [13]–[16] and tube-based stochastic MPC [17]–[22], as well as tube-based distributionally robust MPC [23]–[25]. Interestingly, most tube-based robust MPC proposals reviewed above were generated from [11], while almost all tube-base stochastic and distributionally robust MPC proposals have adopted method [12]. The main differences concern how the optimal control problem (OCP) is solved where the OCP of [11] is solved based on the current state of the nominal system, while [12] solves the OCP based on the current state of the real system. Another notable difference lies in how they tighten the stage constraints where [11] uses (minimal) robust positively invariant sets, while [12] uses a sequence of reachable sets. Compared to [12], recursive feasibility and stability are easier to be obtained in [11], as argued in [26].

However, extending [11] to linear stochastic systems requires computing a probabilistic positively invariant set [27] which is non-trivial. In this regard, [27] proposes a method to compute polyhedral probabilistic positively invariant sets by using Chebyshev inequality, assuming that the system matrix is diagonalisable and that the disturbance distribution has zero-mean and diagonal covariance matrix. The paper [28] proposes to characterize ellipsoidal probabilistic positively invariant sets, assuming that the system matrix is invertible and that the disturbance distribution has zero-mean. In [29], the ellipsoidal probabilistic positively invariant sets are investigated, assuming a correlation bound for the system matrix.

In this chapter, we generalize the method of [11] to linear systems under possibly unbounded stochastic disturbance, where stage-wise chance constraints on the state and input are considered. The proposed MPC only requires knowledge of the mean and covariance of the stochastic disturbance without assuming zero-mean, invertible/diagonal system matrices or bounds on the mean and covariance of the disturbance like in [27]–[29]. To satisfy the stage-wise chance constraints through constraint tightening, a polytopic (minimal) probabilistic positively invariant set is computed, which is minimal for a group of polytopes that are probabilistic positively invariant. The resulting online optimization problem is a simple, standard strictly convex quadratic programming, for which recursive feasibility and stability properties can be easily guaranteed.

The rest of this chapter is organized as follows. Section 4.2 details the problem setup on the systems and constraints. Section 4.3 discusses the probabilistic positively invariant set and its computation. Section 4.4 presents the proposed MPC algorithm. Section 4.5 demonstrates the proposed approach with a numerical case study.Section 4.6 ends this chapter with concluding remarks.

4.1.1 Basic Nomenclature and Conventions

The sets of real numbers, non-negative integers and positive integers are denoted by $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{N}_+$, respectively. Given $a, b \in \mathbb{N}$, with $a < b$, we use the notation $\mathbb{I}_a^b$ to denote the set of non-negative integers $\{a, a+1, \ldots, b\}$. The Minkowski sum and Pontryagin difference of nonempty sets $\mathcal{X}$ and $\mathcal{Y}$ are

$$\begin{aligned}&\mathcal{X} \oplus \mathcal{Y} := \{x + y \,:\, x \in \mathcal{X},\ y \in \mathcal{Y}\} \text{ and}\\ &\mathcal{X} \ominus \mathcal{Y} := \{x \,:\, \forall y \in \mathcal{Y},\ x + y \in \mathcal{X}\},\end{aligned}$$

respectively. The image of a nonempty set $\mathcal{X}$ under a matrix of compatible dimensions M is given by $M\mathcal{X} := \{Mx \,:\, x \in \mathcal{X}\}$. Likewise, if M is a square matrix, for any integer $k \in \mathbb{N}$, $M^k\mathcal{X} := \left\{M^k x \,:\, x \in \mathcal{X}\right\}$. The sets of symmetric positive semi-definite and symmetric positive definite matrices in $\mathbb{R}^n$ are denoted by $\mathbb{S}_+^n$ and $\mathbb{S}_{++}^n$, respectively. Ellipsoids with shape matrix $M \in \mathbb{S}_+^n$, center $c \in \mathbb{R}^n$ and radius $\sqrt{r} > 0$ are defined as

$$\begin{aligned}\mathcal{E}(M, c, r) :=& \{x \,:\, (x-c)^\top M^{-1}(x-c) \le r\}\\ =& \{c + M^{\frac{1}{2}}x \,:\, x^\top x \le r\},\end{aligned}$$

where the matrix $M^{\frac{1}{2}}$ is such that $M^{\frac{1}{2}}M^{\frac{1}{2}} = M$. In addition, it holds that $\mathcal{E}(M, c, r) = c \oplus \mathcal{E}(M, 0, r)$ by definition. The support function $\mathrm{h}(\mathcal{X}, \cdot)$ of a nonempty, closed, convex set $\mathcal{X} \subseteq \mathbb{R}^n$ is given, for all $y \in \mathbb{R}^n$, by

$$\mathrm{h}\left(\mathcal{X}, y\right) := \sup_x \{y^\top x \,:\, x \in \mathcal{X}\},$$

4.2 Problem Setup

This section introduces the considered systems and the underlying constraints and presents the utilized state and control parameterization. This section also includes the standing assumptions on the systems and constraints as well as the parameterization strategy.

4.2.1 Uncertain Linear Systems

This chapter considers uncertain linear systems of the form

$$\forall k \in \mathbb{N}, \quad x_{k+1} = Ax_k + Bu_k + w_k, \tag{4.1}$$

where $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$ denote the state and input, respectively, while $w_k \in \mathbb{R}^n$ denotes a randomly distributed disturbance. These disturbances $w_0, w_1, \ldots : \mathbb{R}^n \to \mathbb{R}^n$ are assumed identically distributed and independent of each other and x_k and u_k, with common probability distribution ω, i.e.,

$$\forall k \in \mathbb{N}, \quad w_k \sim \omega.$$

Due to the disturbance w_k, we consider stage-wise chance constraints on the state x_k and input u_k, given by:

$$\forall k \in \mathbb{N}_+, \quad \Pr(x_k \in \mathcal{X} \mid x_0) \geq 1 - \epsilon_x \text{ and} \tag{4.2}$$

$$\forall k \in \mathbb{N}, \quad \Pr(u_k \in \mathcal{U}) \geq 1 - \epsilon_u, \tag{4.3}$$

where $\epsilon_x, \epsilon_u \in (0, 1)$ are modeling parameters, x_0 is an initial state, and $\mathcal{X}$ and $\mathcal{U}$ denote constraint sets on the state and input respectively. The set $\mathcal{X}$ ($\mathcal{U}$) is a region for which the system state x_k (input u_k) should be confined with a probability no lower than the specified confidence level $1 - \epsilon_x$ ($1 - \epsilon_u$).

We make the following standing assumptions on the system dynamics (4.1) and constraints (4.2)-(4.3)

Assumption 4.1

1. *The matrix pair $(A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$ is known exactly, and it is strictly stabilizable.*

2. *The set $\mathcal{X} \subseteq \mathbb{R}^n$ is a convex polyhedron that contains the origin in its interior, and the set $\mathcal{U} \subseteq \mathbb{R}^m$ is a convex polytope that contains the origin in its interior. In addition, their irredundant inequalities characterizations are given by:*

$$\mathcal{X} := \left\{ x \;:\; \forall i \in \mathcal{I}_{\mathcal{X}},\; f_i^\top x \leq 1 \right\}, \tag{4.4}$$

$$\mathcal{U} := \left\{ u \;:\; \forall i \in \mathcal{I}_{\mathcal{U}},\; g_i^\top u \leq 1 \right\}, \tag{4.5}$$

with the known finite index sets $\mathcal{I}_{\mathcal{X}} := \{1, 2, \ldots, n_{\mathcal{X}}\}$, $\mathcal{I}_{\mathcal{U}} := \{1, 2, \ldots, n_{\mathcal{U}}\}$, and the known vectors $f_i \in \mathbb{R}^n$ and $g_i \in \mathbb{R}^n$.

3. *The probability distribution ω is not known exactly, however, its mean vector $\mu_\omega \in \mathbb{R}^n$ and covariance matrix $\Sigma_\omega \in \mathbb{S}^n_{++}$ are known.*

4.2.2 State and Control Parameterization

In what follows, we introduce the linear feedback law as

$$\forall k \in \mathbb{N}, \quad u_k = v_k + K s_k \tag{4.6}$$

and rewrite the state x_k as

$$\forall k \in \mathbb{N}, \quad x_k = z_k + s_k. \tag{4.7}$$

Here, $z_k \in \mathbb{R}^n$ denotes the nominal (disturbance-free) component such that

$$\forall k \in \mathbb{N}, \quad z_{k+1} = A z_k + B v_k \tag{4.8}$$

based on the nominal control input $v_k \in \mathbb{R}^m$. Thus, by subtracting (4.8) from (4.1) and in view of (4.6)–(4.7), $s_k \in \mathbb{R}^n$ denotes the error component satisfying

$$\forall k \in \mathbb{N}, \quad s_{k+1} = (A + BK) s_k + w_k, \text{ with } w_k \sim \omega. \tag{4.9}$$

The matrix $K \in \mathbb{R}^{m \times n}$ is *a priori* given and constructed to satisfy the following assumption.

Assumption 4.2 *The matrix $K \in \mathbb{R}^{m \times n}$ is given, and it is such that the matrix $A_K := A + BK$ is strictly stable, i.e., all eigenvalues of A_K are in the open unit disc.*

4.3 Probabilistic Positively Invariant Set

In this section, we first characterize the confidence regions for the random vector s_k. Then, sufficient conditions for probabilistic positively invariant sets are presented, which is analogous to characterization of the well understood robust positively invariant sets. A method for computing the polytopic probabilistic invariant set, that is minimal with respect to a group of pre-defined norm vectors, is presented based on linear programming.

4.3.1 Characterization Confidence Regions

Let $\mu_k \in \mathbb{R}^n$ and $\Sigma_k \in \mathbb{S}^n_{++}$ denote the mean vector and covariance matrix of the random vector s_k, whose stochasticity is caused by the random disturbance sequence $\{w_i\}_{i=0}^{k-1}$ via stochastic dynamics (4.9). Then, the dynamics of μ_k and Σ_k are specified, with $\mu_0 = s_0$ and $\Sigma_0 = 0$, by

$$\forall k \in \mathbb{N}, \quad \mu_{k+1} = A_K \mu_k + \mu_\omega \text{ and} \tag{4.10}$$

$$\forall k \in \mathbb{N}, \quad \Sigma_{k+1} = A_K \Sigma_k A_K^\top + \Sigma_\omega. \tag{4.11}$$

Based on the multivariate Chebyshev inequality [30], [31], we have the following proposition.

Proposition 4.1 *Suppose Assumption 4.1–3 holds. Consider μ_k and Σ_k generated by* (4.10) *and* (4.11)*. It follows that*

$$\forall k \in \mathbb{N}_+, \quad \Pr\left(s_k \in \mathcal{E}(\Sigma_k, \mu_k, n/\epsilon) \mid s_0\right) \geq 1 - \epsilon, \tag{4.12}$$

i.e., $\mathcal{E}(\Sigma_k, \mu_k, n/\epsilon)$ are confidence regions with confidence level $1-\epsilon$ of the random vectors s_k.

Proof: It follows from [30, Theorem 1] that

$$\begin{aligned}\forall k \in \mathbb{N}_+, \quad & \Pr\left((s_k - \mu_k)^\top \Sigma_k^{-1}(s_k - \mu_k) > n/\epsilon \mid s_0\right) \\ & \leq \Pr\left((s_k - \mu_k)^\top \Sigma_k^{-1}(s_k - \mu_k) \geq n/\epsilon \mid s_0\right) \\ & \leq \epsilon,\end{aligned}$$

such that

$$\begin{aligned}\forall k \in \mathbb{N}_+, \; & \Pr\left((s_k - \mu_k)^\top \Sigma_k^{-1}(s_k - \mu_k) \leq n/\epsilon \mid s_0\right) \\ & = 1 - \Pr\left((s_k - \mu_k)^\top \Sigma_k^{-1}(s_k - \mu_k) > n/\epsilon \mid s_0\right) \\ & \geq 1 - \epsilon.\end{aligned}$$

Thus, it follows from the definition of ellipsoids that

$$\forall k \in \mathbb{N}_+, \quad \Pr\left(s_k \in \mathcal{E}(\Sigma_k, \mu_k, n/\epsilon) \mid s_0\right) \geq 1 - \epsilon.$$

□

Corollary 4.1 *Suppose Assumptions 4.1 and 4.2 hold. Consider μ_k and Σ_k generated by* (4.10) *and* (4.11). *It follows that*

$$\forall k \in \mathbb{N}_+, \quad \mathcal{E}(\Sigma_{k+1}, \mu_{k+1}, n/\epsilon) \subseteq A_K \mathcal{E}(\Sigma_k, \mu_k, n/\epsilon) \oplus \mathcal{E}(\Sigma_\omega, \mu_\omega, n/\epsilon). \tag{4.13}$$

Proof: By the property of ellipsoids, it holds that

$$\forall k \in \mathbb{N}_+, \quad \mathcal{E}(\Sigma_{k+1}, \mu_{k+1}, n/\epsilon) = \mu_{k+1} \oplus \mathcal{E}(\Sigma_{k+1}, 0, n/\epsilon)$$

and

$$\begin{aligned} &A_K \mathcal{E}(\Sigma_k, \mu_k, n/\epsilon) \oplus \mathcal{E}(\Sigma_\omega, \mu_\omega, n/\epsilon) \\ &= A_K \mu_k + \mu_\omega \oplus A_K \mathcal{E}(\Sigma_k, 0, n/\epsilon) \oplus \mathcal{E}(\Sigma_\omega, 0, n/\epsilon). \end{aligned}$$

For all $k \in \mathbb{N}_+$, it follows from [29, Property 1] that

$$\mathcal{E}(\Sigma_{k+1}, 0, n/\epsilon) \subseteq A_K \mathcal{E}(\Sigma_k, 0, n/\epsilon) \oplus \mathcal{E}(\Sigma_\omega, \mu_\omega, n/\epsilon),$$

and it follows from (4.10) that

$$\mu_{k+1} = A_K \mu_k + \mu_\omega.$$

Thus, the Minkowski sum of both sides of the above two expressions completes the proof. □

4.3.2 Sufficient Conditions for Probabilistic Positively Invariant Sets

At this point, we introduce the sets $\mathcal{R}_k^\epsilon$ defined by the following set-dynamics:

$$\forall k \in \mathbb{N}, \quad \mathcal{R}_{k+1}^\epsilon = A_K \mathcal{R}_k^\epsilon \oplus \mathcal{E}(\Sigma_\omega, \mu_\omega, n/\epsilon) \tag{4.14}$$

with the boundary condition given by $\mathcal{R}_0^\epsilon = \{s_0\}$.

Proposition 4.2 *Suppose Assumptions 4.1 and 4.2 hold. Consider the sets $\mathcal{R}_k^\epsilon$ generated by* (4.14) *and μ_k and Σ_k generated by* (4.10) *and* (4.11). *It follows that*

$$\forall k \in \mathbb{N}_+, \quad \mathcal{E}(\Sigma_k, \mu_k, n/\epsilon) \subseteq \mathcal{R}_k^\epsilon, \tag{4.15}$$

such that

$$\begin{aligned} \forall k \in \mathbb{N}_+, \quad \Pr(s_k \in \mathcal{R}_k^\epsilon \mid s_0) &\geq \Pr(s_k \in \mathcal{E}(\Sigma_k, \mu_k, n/\epsilon) \mid s_0) \\ &\geq 1 - \epsilon. \end{aligned} \tag{4.16}$$

Proof: We will show by induction that the inclusions (4.15) holds. When $k = 1$, we have that

$$\begin{aligned}\mathcal{E}(\Sigma_1, \mu_1, n/\epsilon) &= A\mu_0 \oplus \mathcal{E}(\Sigma_\omega, \mu_\omega, n/\epsilon)\\ \mathcal{R}_1^\epsilon &= As_0 \oplus \mathcal{E}(\Sigma_\omega, \mu_\omega, n/\epsilon)\end{aligned}$$

such that $\mathcal{E}(\Sigma_1, \mu_1, n/\epsilon) = \mathcal{R}_1^\epsilon \subseteq \mathcal{R}_1^\epsilon$ since $\mu_0 = s_0$. Suppose, for some $k \in \mathbb{N}_+$, $\mathcal{E}(\Sigma_k, \mu_k, n/\epsilon) \subseteq \mathcal{R}_k^\epsilon$, then we have that

$$\begin{aligned}\mathcal{E}(\Sigma_{k+1}, \mu_{k+1}, n/\epsilon) &\subseteq A_K\mathcal{E}(\Sigma_k, \mu_k, n/\epsilon) \oplus \mathcal{E}(\Sigma_\omega, \mu_\omega, n/\epsilon)\\ &\subseteq A_K\mathcal{R}_k^\epsilon \oplus \mathcal{E}(\Sigma_\omega, \mu_\omega, n/\epsilon)\\ &= \mathcal{R}_{k+1}^\epsilon,\end{aligned}$$

in which the first inclusion follows from (4.13). It follows from (4.12) and (4.15) that (4.16) holds. □

We recall from [27] that a set $\mathcal{S} \subseteq \mathbb{R}^n$ is called probabilistic positively invariant for the system (4.9) with probability $1-\epsilon$, if and only if, for any $s_0 \in \mathcal{S}$, $\Pr(s_k \in \mathcal{S} \mid s_0) \geq 1-\epsilon$ for all $k \in \mathbb{N}_+$.

Proposition 4.3 *Suppose Assumptions 4.1 and 4.2 hold. Consider a set $\mathcal{R}^\epsilon \subseteq \mathbb{R}^n$ satisfying, for any $s_0 \in \mathcal{R}^\epsilon$,*

$$A_K\mathcal{R}^\epsilon \oplus \mathcal{E}(\Sigma_\omega, \mu_\omega, n/\epsilon) \subseteq \mathcal{R}^\epsilon \tag{4.17}$$

with $\epsilon \in (0, 1)$. Then, $\mathcal{R}^\epsilon$ is a probabilistic positively invariant set for the system (4.9) with probability $1 - \epsilon$.

Proof: Consider the sets $\mathcal{R}_k^\epsilon$ generated by (4.14). As shown in (4.16), we have that

$$\forall k \in \mathbb{N}_+, \quad \Pr(s_k \in \mathcal{R}_k^\epsilon \mid s_0) \geq 1 - \epsilon.$$

To show that $\mathcal{R}^\epsilon$ is a probabilistic positively invariant set for the stochastic system (4.9) with probability $1 - \epsilon$, we just need to prove that for all $s_0 \in \mathcal{R}^\epsilon$ it holds that $\mathcal{R}_k^\epsilon \subseteq \mathcal{R}^\epsilon$, $k \in \mathbb{N}$. When $k = 0$,

$$\mathcal{R}_0^\epsilon = \{s_0\} \subseteq \mathcal{R}^\epsilon$$

trivially holds. Suppose, for some $k \in \mathbb{N}$, $\mathcal{R}_k^\epsilon \subseteq \mathcal{R}^\epsilon$, then we have that

$$\begin{aligned}\mathcal{R}_{k+1}^\epsilon &= A_K \mathcal{R}_k^\epsilon \oplus \mathcal{E}(\Sigma_\omega, \mu_\omega, n/\epsilon)\\ &\subseteq A_K \mathcal{R}^\epsilon \oplus \mathcal{E}(\Sigma_\omega, \mu_\omega, n/\epsilon)\\ &\subseteq \mathcal{R}^\epsilon.\end{aligned}$$

Thus, we have, for all $s_0 \in \mathcal{R}^\epsilon$, that $\mathcal{R}_k^\epsilon \subseteq \mathcal{R}^\epsilon,\ k \in \mathbb{N}$, by induction. □

In the rest of this section, we make the following standing assumption

Assumption 4.3 *Given $\epsilon \in (0,1)$, the set $\mathcal{E}(\Sigma_\omega, \mu_\omega, n/\epsilon)$ contains the origin in its interior.*

Note that Assumption 4.3 implies that the support functions of $\mathcal{E}(\Sigma_\omega, \mu_\omega, n/\epsilon)$ is positive, i.e., for all $y \in \mathbb{R}^n$, $\mathrm{h}\left(\mathcal{E}(\Sigma_\omega, \mu_\omega, n/\epsilon), y\right) > 0$. We will discuss the methods for computing a probabilistic positively invariant set $\mathcal{R}^\epsilon$ based on the sufficient condition given by (4.17).

4.3.3 Computation of Probabilistic Positively Invariant Sets

Given $\epsilon \in (0,1)$ satisfying Assumption 4.3, we recall that a set $\mathcal{R}^\epsilon \subseteq \mathbb{R}^n$ is robust positively invariant for the uncertain dynamics $\forall k \in \mathbb{N},\ s_{k+1} = A_K s_k + w_k$ with $w_k \in \mathcal{E}(\Sigma_\omega, \mu_\omega, n/\epsilon)$ if and only if

$$A_K \mathcal{R}^\epsilon \oplus \mathcal{E}(\Sigma_\omega, \mu_\omega, n/\epsilon) \subseteq \mathcal{R}^\epsilon, \tag{4.18}$$

and it is the minimal robust positively invariant set if and only if

$$A_K \mathcal{R}^\epsilon \oplus \mathcal{E}(\Sigma_\omega, \mu_\omega, n/\epsilon) = \mathcal{R}^\epsilon. \tag{4.19}$$

Thus, condition (4.17) is equivalent to the sufficient and necessary condition for characterizing robust positively invariant sets for the system $\forall k \in \mathbb{N},\ s_{k+1} = A_K s_k + w_k$ with $w_k \in \mathcal{E}(\Sigma_\omega, \mu_\omega, n/\epsilon)$. Moreover, the minimal robust positively invariant set given by (4.19) is desirable since it will be used to tighten the constraint set in the next section.

However, computing an exact representation of the minimal robust positively invariant set is generally impossible. In practice, outer invariant approximations of the minimal robust positively invariant set are computed [32]. In our setting, since $\mathcal{E}(\Sigma_\omega, \mu_\omega, n/\epsilon)$ is an ellipsoid rather than a polytope, the method [32] is not applicable. Instead, we make use of the method [33], which is generated from the method originally proposed in [16]. The method of [33] computes a polytopic robust positively invariant set that is minimal with respect to the group of robust positively invariant sets generated from a finite number of inequalities with pre-defined normal vectors. In the sequel, we briefly summarize the method of [33]. Consider a set $\mathcal{R}^\epsilon(q)$ given, with $\mathcal{I}_\mathcal{P} = \{1, ..., r\}$, $r \in \mathbb{N}_+$, by

$$\mathcal{R}^\epsilon(q) = \{x \in \mathbb{R}^n : \forall i \in \mathcal{I}_\mathcal{P},\ p_i^\top x \le q_i\}, \tag{4.20}$$

where $\{p_i : \forall i \in \mathcal{I}_\mathcal{P}\}$ spans $\mathbb{R}^n$, $\forall i \in \mathcal{I}_\mathcal{P}$, $q_i \ge 0$ and $q = (q_1, \dots, q_r)^\top$. Then, the polytopic, minimal robust positively invariant set generated from the pre-defined normal vectors $\{p_i : \forall i \in \mathcal{I}_\mathcal{P}\}$ is given by

$$\mathcal{R}^\epsilon(q^*) = \{x \in \mathbb{R}^n : \forall i \in \mathcal{I}_\mathcal{P},\ p_i^\top x \le q_i^*\}. \tag{4.21}$$

Here, $q^* = d^* + c^*$, where $d^* := (d_1^*, ..., d_r^*)^\top$ is given by

$$\begin{aligned} \forall i \in \mathcal{I}_\mathcal{P}, \quad d_i^* &= \mathrm{h}\left(\mathcal{E}(\Sigma_\omega, \mu_\omega, n/\epsilon), p_i\right) \\ &= \mu_\omega^\top p_i + \sqrt{\frac{n(p_i^\top \Sigma_\omega p_i)}{\epsilon}}, \end{aligned} \tag{4.22}$$

and $c^* := (c_1^*, \dots, c_r^*)^\top$ is given by the solution to the following linear programming problem:

$$\begin{aligned} &\max_{c,\,\xi} \ \sum_{i=0}^{r} c_i \\ &\text{s.t.} \ \begin{cases} \forall i \in \mathcal{I}_\mathcal{P}, \quad c_i \le p_i^\top A_K \xi_i \\ \forall i \in \mathcal{I}_\mathcal{P},\ \forall j \in \mathcal{I}_\mathcal{P}, \quad p_j^\top \xi_i \le c_i + d_i^* \end{cases} \end{aligned} \tag{4.23}$$

with $c := (c_1, \dots, c_r)^\top$ and $\xi := \{\xi_i \in \mathbb{R}^n\}_{i=1}^r$. Note that in [33], c^* and d^* are computed all together in one optimization problem, but d^* could be also pre-computed as commented in Remark 4 of [33]. Therefore, in view of Proposition 4.3, $\mathcal{R}^\epsilon(q^*)$ is also a probabilistic positively invariant set for the system (4.9) with probability no less than $1 - \epsilon$.

4.4 Robustifying MPC In Probabilistic Ways

This section presents the main formulation of the robustifying MPC controller and analyzes the control-theoretic properties of the resulting MPC controlled system.

4.4.1 Robustifying MPC Formulation

Within our setting, given $N \in \mathbb{N}_+$, for any nominal state $z_k \in \mathbb{R}^n$ at the sampling time $k \in \mathbb{N}$, the proposed MPC solves an OCP of the form:

$$V_N^0(z_k) = \min_{\mathbf{z}_k,\, \mathbf{v}_k} \sum_{t=0}^{N-1} \ell(z_{t|k}, v_{t|k}) + V_f(z_{N|k})$$
$$\text{s.t.} \begin{cases} \forall t \in \mathbb{I}_0^{N-1},\ z_{t+1|k} = A z_{t|k} + B v_{t|k}, \\ \forall t \in \mathbb{I}_1^{N-1},\ z_{t|k} \in \mathcal{Z}, \\ \forall t \in \mathbb{I}_0^{N-1},\ v_{t|k} \in \mathcal{V}, \\ z_{N|k} \in \mathcal{Z}_f,\ z_{0|k} = z_k, \end{cases} \tag{4.24}$$

with $\mathbf{z}_k := (z_{0|k}^\top, ..., z_{N|k}^\top)^\top$ and $\mathbf{v}_k := (v_{0|k}^\top, ..., v_{N-1|k}^\top)^\top$. The subscript $t|k$ denotes the predictions of the nominal state and control t-steps ahead of the sampling time k. The stage and terminal costs are given by

$$\ell(z, v) = z^\top Q z + v^\top R v \quad \text{and} \quad V_f(z) = z^\top P z\ , \tag{4.25}$$

for all $z \in \mathbb{R}^n$ and all $v \in \mathbb{R}^m$, with $P, Q \in \mathbb{S}_{++}^n$ and $R \in \mathbb{S}_{++}^m$. The stage constraint sets on the nominal state and input are given by

$$\mathcal{Z} := \mathcal{X} \ominus \mathcal{R}^{\epsilon_x}(q^*) \quad \text{and} \quad \mathcal{V} := \mathcal{U} \ominus K\mathcal{R}^{\epsilon_u}(q^*). \tag{4.26}$$

Assumption 4.4 *The nominal state constraint set $\mathcal{Z}$ is a nonempty convex polyhedron that contains the origin in its interior, and the nominal input constraint set $\mathcal{V}$ is a nonempty convex polytope that contains the origin in its interior.*

This assumption implies that $\mathcal{R}^{\epsilon_x}(q^*) \subseteq \text{interior}(\mathcal{X})$ and $K\mathcal{R}^{\epsilon_u}(q^*) \subseteq \text{interior}(\mathcal{U})$, i.e.,

$$\forall i \in \mathcal{I}_{\mathcal{X}}, \quad \mathrm{h}\left(\mathcal{R}^{\epsilon_x}(q^*), f_i\right) < 1 \text{ and}$$
$$\forall i \in \mathcal{I}_{\mathcal{U}}, \quad \mathrm{h}\left(\mathcal{R}^{\epsilon_u}(q^*), K^\top g_i\right) < 1.$$

The sets $\mathcal{Z}$ and $\mathcal{V}$ can be efficiently computed by using support functions given by

$$\mathcal{Z} := \left\{ z \ :\ \forall i \in \mathcal{I}_{\mathcal{X}},\ f_i^\top z \leq 1 - \mathrm{h}\left(\mathcal{R}^{\epsilon_x}(q^*), f_i\right) \right\},$$
$$\mathcal{V} := \left\{ v \ :\ \forall i \in \mathcal{I}_{\mathcal{U}},\ g_i^\top v \leq 1 - \mathrm{h}\left(\mathcal{R}^{\epsilon_u}(q^*), K^\top g_i\right) \right\}.$$

The nominal terminal constraint set $\mathcal{Z}_f$ and cost V_f satisfy the following natural conditions [5].

Assumption 4.5

1. *Let $K_f \in \mathbb{R}^{n\times m}$ be such that $A_{K_f} := A + BK_f$ is strictly stabilizable. The set $\mathcal{Z}_f$ is a convex polyhedron, and it is the maximal positively invariant set for the system $z^+ = A_{K_f} z$ with constraints $z \in \mathcal{Z}$ and $K_f z \in \mathcal{V}$, i.e., it is the maximal set such that*

$$A_{K_f}\mathcal{Z}_f \subseteq \mathcal{Z}_f, \quad \mathcal{Z}_f \subseteq \mathcal{Z} \quad \textit{and} \quad K_f\mathcal{Z}_f \subseteq \mathcal{V}. \tag{4.27}$$

2. *It holds that, for all $z \in \mathcal{Z}_f$,*

$$V_f((A + BK_f)z) + \ell(z, K_f z) \leq V_f(z). \tag{4.28}$$

We define the feasible set for $N \in \mathbb{N}_+$ by

$$\mathbb{X}_N := \{z_k \in \mathbb{R}^n : z_k \text{ is such that (4.24) is feasible}\}.$$

In our setting, the optimization problem (4.24) is a convex quadratic programming problem, feasible for all $z_k \in \mathbb{X}_N$. We denote the parametric solution map of (4.24) by $\mathbf{z}_k^0(z_k)$ and $\mathbf{v}_k^0(z_k)$. Finally, we denote the MPC feedback law on the nominal system (4.8) and the real system (4.1) by

$$\begin{aligned}&\bar{\kappa}_N(z_k) = v_{0|k}^0(z_0) \text{ and}\\&\kappa_N(x_k) = \bar{\kappa}_N(z_k) + K(x_k - z_k),\end{aligned}$$

respectively, such that the closed-loop controlled systems are given by

$$\forall k \in \mathbb{N}, \quad z_{k+1} = Az_k + B\bar{\kappa}_N(z_k) \text{ and} \tag{4.29}$$

$$\forall k \in \mathbb{N}, \quad x_{k+1} = Ax_k + B\kappa_N(x_k) + w_k,\ w_k \sim \omega. \tag{4.30}$$

4.4.2 Control-Theoretic Properties

The proposed MPC scheme is briefly summarized in the Algorithm 1. By construction, we have that $z_k \in \mathcal{Z}$ for all $k \in \mathbb{N}_+$, $v_k \in \mathcal{V}$ for all $k \in \mathbb{N}$, and the sets $\mathcal{R}^{\epsilon_x}(q^*)$ and $\mathcal{R}^{\epsilon_u}(q^*)$ are probabilistic positively invariant sets with probabilities $1 - \epsilon_x$ and $1 - \epsilon_x$, respectively. Thus, the condition $x_0 - z_0 \in \mathcal{R}^{\epsilon_x}(q^*)$ and the state and input decomposition (4.6)–(4.7) ensure that chances constraints (4.2) and (4.3) are satisfied.

Algorithm 1 Robustifing model predictive control of chance constrained linear systems

Initialization: Given an initial system state x_0, selecting an initial nominal state z_0 such that $x_0 - z_0 \in \mathcal{R}^{\epsilon_x}(q^*)$.

for all $k = 0 \to \infty$, **do**

1. Solve the OCP (4.24) to obtain $\mathbf{z}_k^0(z_k)$ and $\mathbf{v}_k^0(z_k)$.
2. Apply the $\bar{\kappa}_N(z_k)$ and $\kappa_N(x_k)$ to (4.29) an (4.30), respectively.
3. Measure the real state x_{k+1} and nominal state z_{k+1} from (4.29) an (4.30), respectively.
4. Set $k \leftarrow k+1$, and go to Step 1.

Theorem 4.1 *Suppose Assumptions 4.1–4.5 hold. Denote ρ_∞ as the stationary process of the system $\forall k \in \mathbb{N},\ s_{k+1} = A_K s_k + w_k$, with $w_k \sim \omega$. For all $z_0 \in \mathbb{X}_N$, the MPC optimization problem* (4.24) *is recursively feasible. The MPC controlled nominal system* (4.29) *is asymptotically stable to the origin. The real state x_k of the MPC controlled system* (4.30) *converges to the stationary process ρ_∞ as $k \to \infty$.*

Proof: The recursive feasibility of (4.24) is guaranteed by the invariant property of the terminal set $\mathcal{Z}_f$ enforced in Assumption 4.5–i, as shown in [4], [5]. Similarly, Assumption 4.5–ii together with the invariance property of $\mathcal{Z}_f$ guarantee a strict decrease of the value function $V_N^0(\cdot)$ along the closed-loop nominal state trajectory [4], [5], which guarantees that the nominal MPC controller $\bar{\kappa}_N(\cdot)$ is asymptotically stabilizing for (4.29). Because A_K is strictly stabilizable, s_k converges to a stationary process ρ_∞ as $k \to \infty$. Since $x_k = z_k + s_k$ and $u_k = v_k + K s_k$, it follows from, $z_k \to 0$ and $v_k \to 0$ as $k \to \infty$, that x_k converges to ρ_∞. □

4.5 Numerical Illustration

We consider a DC-DC converter model that has previously been adopted in [19], [20], and it is of the form

$$\forall k \in \mathbb{N},\ x_{k+1} = \begin{bmatrix} 1 & 0.075 \\ -1.43 & 0.996 \end{bmatrix} x_k + \begin{bmatrix} 4.798 \\ 0.115 \end{bmatrix} u_k + w_k,$$

where the mean vector and covariance matrix of the stochastic disturbance $w_k \sim \omega$ are given by

$$\mu_\omega = \begin{bmatrix} 0.005 \\ 0.005 \end{bmatrix} \quad \text{and} \quad \Sigma_\omega = 10^{-4} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.31}$$

The state and input constraint sets are given by

$$\begin{aligned} \mathcal{X} &:= \left\{ x \in \mathbb{R}^2 \;\middle|\; -2 \le [1\ 0]x \le 2,\ -3 \le [0\ 1]x \le 3 \right\} \\ \mathcal{U} &:= \{ u \in \mathbb{R} \mid -0.4 \le u \le 0.4 \}. \end{aligned}$$

The weighting matrices for the stage and terminal costs are given by

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}, \ R = 1 \text{ and } P = \begin{bmatrix} 1.9074 & -5.0562 \\ -5.0562 & 39.5448 \end{bmatrix}.$$

For the sake of simplicity, the control feedback and local terminal feedback matrices are specified by

$$K = K_f = [-0.2858\ \ 0.4910],$$

such that the matrices A_K and A_{K_f} are strictly stabilizable. Note that K and K_f are not required to be the same. The terminal weighting matrix P and the feedback matrix K_f are obtained as the solutions to the infinite horizon unconstrained optimal control for (A, B, Q, R). The modeling parameters in stage-wise chance constraints (4.2)-(4.3) are specified by

$$\epsilon_x = \epsilon_u = 0.2.$$

To compute a polytopic probabilistic positively invariant set given by (4.21), we define the normal vectors $p_i \in \mathbb{R}^2$ as

$$\forall i \in \{1, ..., r\}, \quad p_i = \left[\sin\left(\tfrac{2\pi(i-1)}{r}\right) \quad \cos\left(\tfrac{2\pi(i-1)}{r}\right) \right]^\top,$$

with $r = 66$. Then, $q^* = c^* + d^*$ is computed from the solutions to (4.22) and (4.23).

We choose a prediction horizon $N = 10$ and start the control process from an initial state $x_0 = [2.6\ \ 3.2]^\top$. For convenience, we set $z_0 = x_0$. Figure 4.1 visualizes the closed-loop real state trajectories and nominal trajectories with some disturbance sequences $\{w_k\}_{k=0}^{25}$ sampled from the multivariate normal distribution whose mean and covariance are given in (4.31). By simulating the closed-loop system with 10^4 different realizations of the disturbance sequence $\{w_k\}_{k=0}^{25}$, we observe that the average state constraint violation in the first nine steps is 2%, while the input constraints were not violated. These results are rather conservative than the pre-defined violating probabilities $\epsilon_x = \epsilon_u = 0.2$. This

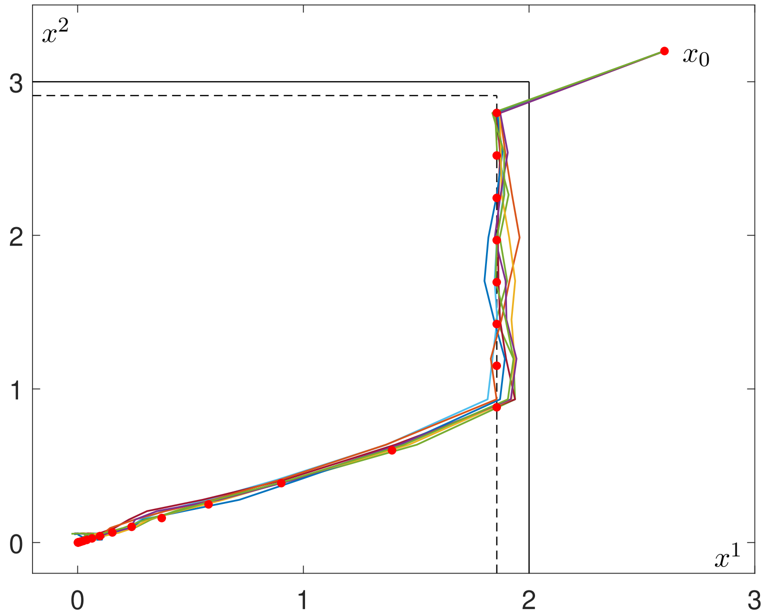

Figure 4.1: The closed-loop real state trajectories of (4.30) with 8 sampled disturbance sequences (coloured lines) and the closed-loop nominal state trajectories of (4.29) (red dots). The dashed and solid lines denote partial borders of $\mathcal{Z}$ and $\mathcal{X}$, respectively.

discrepancy is due to the fact that the computed probabilistic positively invariant sets are conservative since only mean and covariance information are used in computing the confidence regions. Figure 4.2 shows the nominal state trajectory of z_k given by (4.29) and the nominal input trajectory of $\bar{\kappa}_N(z_k)$, illustrating that the controlled nominal system asymptotically convergences to the origin. Consequently, the real state x_k of (4.30) converges to the stationary process, and the set $\mathcal{R}_x^{\epsilon}(q^*)$ is confidence region with probability $1 - \epsilon_x$ of this stationary process.

4.6 Concluding Remarks

This chapter has introduced a model predictive controller that provides sufficient robustness to reject possibly unbounded stochastic disturbances with stage-wise chance constraints on the system state and input being guaranteed. The online computation reduces to a simple standard quadratic quadratic programming problem. Compared to conventional MPC algorithms, the proposed method only needs minor offline computational efforts to compute a probabilistic positively invariant set, which can be easily

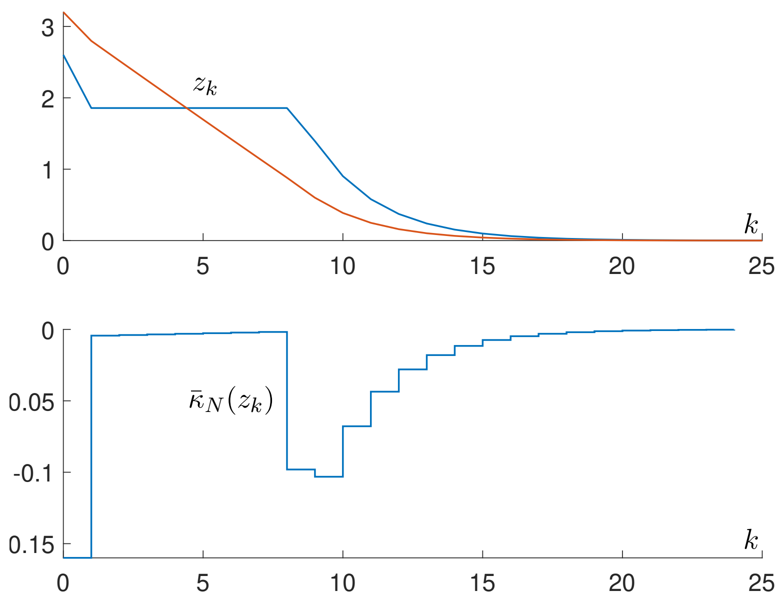

Figure 4.2: The closed-loop nominal state z_k (upper part) and the closed-loop nominal input $\bar{\kappa}_N(z_k)$ (lower part).

computed by solving a simple linear programming problem. Future work will focus on mitigating conservativeness and extending the utilized certainty-equivalent cost function to expected ones.

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Chapter 5

Stochastic Linear MPC with Sound Control-Theoretic Properties

This Chapter is based on the following paper under review:

Kai Wang, Oliver K. Hasler and Sébastien Gros. Stochastic MPC with Robust Positive Invariance and Exponential Stability Guarantees. *Submitted to IEEE Transactions on Automatic Control*, 2024.

This Chapter is awaiting publication and is not included in NTNU Open

Chapter 6

Mission-Wide Chance-Constrained Optimal Control via DP

This Chapter is based on the following paper:

Kai Wang and Sébastien Gros. Solving Mission-Wide Chance-Constrained Optimal Control Using Dynamic Programming. In *Proceedings of the 61st IEEE Conference on Decision and Control (CDC)*, 2022.

6.1 Introduction

In real-world control applications uncertainties, plant-model mismatch and exogenous disturbances are unavoidable. Some finite-horizon Markov control problems require that a risk criterion must be met over the entire planning horizon. Such a requirement could arise due to certain regulations, such as the safety-stock placement in supply chain management and the collision-avoidance in robot path planning. Performing optimal control of such system under hard (robust) constraints can yield very conservative control policies or even be infeasible. Another commonly used approach is to add a penalty term to the objective function. However, it is hard to decided how strict the penalty should be set; high penalties may result in conservative solutions, while low penalties may result in high risk. It is therefore common to rather handle the risk using chance (probabilistic) constraints directly.

In the literature of optimal control, there are many forms of chance constraint. For example, in path planning for vehicles in the presence of obstacles, the mission is supposed to plan an optimal trajectory for a vehicle within N time stages. Suppose that there exist m constraints for the vehicle at each state and time stage. *Individual chance constraint* restricts the probability that the state violates a single constraint at a single time stage. *Stage-wise chance constraint* restricts the probability that the state violates any individual constraints at a single time stage, which consists of m individual chance constraints. Both of them in effect restrict at every time stage the probability that the vehicle collides with an obstacle. *Mission-wide chance constraint* restricts the probability that the state sequence/trajectory spanning over all time steps violates any constraints among the total mN constraints. In contrast to individual or stage-wise ones, a mission-wide chance constraint directly restricts the probability of collision on the overall driving mission. A mission-wide chance constraint is arguably more meaningful than stage-wise constraints in some specific control tasks. Indeed, the former directly handles the risk of running a mission [1]–[3], while the latter does it very indirectly since satisfying a risk level in each time step may result in a poor risk level over the entire horizon, see Section II-A of [4] for detailed explanations. However, individual or stage-wise chance constraints are easier to handle than mission-wide constraints. Indeed, the mission-wide chance constraints involve probabilities over entire state trajectory, yielding very large probability spaces.

Notice that there is another widely used terminology, *joint chance constraint*, referring to the stage-wise constraint in e.g. [5], as well as the mission-wide constraint in e.g. [1], and even the conjunction of the stage-wise constraints within only the prediction horizon that is typically shorter than the mission duration in some stochastic Model Predictive Control (MPC) literature, e.g., [6]. In this chapter we do not adopt this terminology to reduce the risk of confusion.

Optimization subject to the chance constraints was first proposed in the seminal work [7]. Chance-constrained optimization problems are typically intractable. Two main reasons behind this intractability are (i) the difficulty of checking the feasibility of a solution as it requires evaluating multivariate integrals, and (ii) the non-convex feasible region defined by a chance constraint.

The current research on Chance-Constrained Optimal Control Problems (CC-OCP) is centered around tractable approximation approaches, such as Convex bounding approaches [2], [6], [8] and sampling approaches [9], [10]. The closed-loop controller, stochastic MPC, has been widely used in practice to implement the CC-OCP and has been intensively investigated among the MPC community, see [11], [12] and references therein. It approximates the optimal control policy sequence and executes its control action online. To the best of our knowledge, however, there does not exist any stochastic

MPC schemes that explicitly guarantee a rigorous satisfaction of the mission-wide chance constraint. [4] provides a tentative stochastic MPC solution that ensures recursive feasibility in a specific sense, while the mission-wide chance constraint satisfaction is conserved.

Stochastic Dynamic Programming (DP) is a general framework for model-based sequential decision-making processes under uncertainty and provides a global optimal control policy sequence. Chance-constrained DP was first introduced in [13] with application in operation of reservoir, and was extended in [14] by introducing additional system variables. In [13], [14], however, these approaches are heuristic and the chance constraints are limited to the form of cumulative stage-wise chance constraints. [15] provided a systematic way to cope with the formulation proposed in [13] within the framework of Lagrangian duality theory, and also pointed out that there is no clear way to incorporate the mission-wide chance constraint because of the lose of additive structure in the constraints. This is due to the fact that there exists time-correlation between the stage-wise chance constraints such that the mission-wide chance constraint cannot be simply expressed as the summation of stage-wise chance constraints. The authors in [16] proposed a DP method for MWCC-OCP, by first reformulating the mission-wide chance constraint via Boole's approximation, and then solving the resulting unconstrained Lagrangian function. This method was applied to the robotic space exploration mission in [3]. When the optimization criteria is absent, in [17], a tailored DP approach was proposed to maximize the mission-wide probability of safety. Unfortunately, MWCC-OCP cannot necessarily be put in that simple form.

The above statement does not exclude the existence of a DP solution to the MWCC-OCP, but it indicates that classical DP framework cannot be simply applied. This chapter entails in theory that there does exist an exact DP solution to the MWCC-OCP through proper state augmentation. This comes at the cost of the augmented state being in a functional space, hence making the resulting DP problem significantly more challenging to tackle.

The rest of this chapter is structured as follows. In Section 6.2 we formulate the MWCC-OCP and provide some preliminaries. Section 6.3 is devoted to develop a DP algorithm for the general risk-constrained dynamic optimization problem. In Section 6.4 we detail a specific DP algorithm to the MWCC-OCP through state augmentation. An one-dimensional lineal case study is given in Section 6.5. Finally, Section 6.6 concludes the chapter.

6.2 Problem Setup and Preliminaries

Consider a discrete-time Markov Decision Process (MDP) with continuous state and action space. More specifically, we assume the state space $\mathcal{S} \subseteq \mathbb{R}^{n_s}$ and control/action space $\mathcal{A} \subseteq \mathbb{R}^{n_a}$. The transition density function

$$\rho[\, s^+ \,|\, s, a \,] : \mathcal{S} \times \mathcal{S} \times \mathcal{A} \to [0, \infty), \tag{6.1}$$

provides a conditional probability of observing a transition from the state-action pair $s \in \mathcal{S}$, $a \in \mathcal{A}$ to a successor state $s^+ \in \mathcal{S}$. In the control community, the system dynamics are often alternatively defined as where $f : \mathcal{S} \times \mathcal{A} \times \mathcal{W} \to \mathcal{S}$ is possibly a vector-valued nonlinear function and the disturbance w is the realization of a random variable that takes values from set $\mathcal{W} \subseteq \mathbb{R}^{n_w}$. We will assume that the finite stage cost is given by $\ell(s, a) : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$, and that deterministic, Markovian policies are functions $\pi_k(s_k) : \mathcal{S} \to \mathcal{A}$, which specify an action $a_k = \pi_k(s_k)$ when the system visits state s_k at time step k.

Let a nonzero $N \in \mathbb{N}$ denote the mission duration. To accomplish a mission, a decision maker or controller has to make actions/decisions that minimize the following cost

$$\mathbb{E}\left[\sum_{k=0}^{N-1} \ell(s_k, a_k) + \ell_N(s_N) \,\middle|\, s_0, a_k = \pi_k(s_k)\right] . \tag{6.2}$$

Here, $\mathbb{E}$ is the expected value operator applying to the space of all possible trajectories $s_{0,\ldots,N}$ of the states in closed-loop with policy sequence $\pi := \{\pi_0, \ldots, \pi_{N-1}\}$ and a fixed initial state $s_0 \in \mathbb{S}$. Function ℓ_N is a finite terminal cost incurred at the end of the mission. We set $\pi^k := \{\pi_k, \ldots, \pi_{N-1}\}$ for any $k = 1, \ldots, N-1$.

We consider a mission to be safe if the closed-loop trajectory of state lies in a constraint set $\mathbb{S} \subseteq \mathcal{S}$, i.e., $s_{1,\ldots,N} \in \mathbb{S}$. However, in many practical cases, satisfying this safety requirement may be impossible especially when the disturbances of the system are unbounded. Even in the case where the system states are finitely supported, it can still be infeasible, and yield very conservative policies. Alternatively, we seek to guarantee a probabilistic safety on the state sequence spanning over the whole mission, i.e., a mission-wide chance constraint (MWCC):

$$\mathbb{P}\left[\, s_{1,\ldots,N} \in \mathbb{S} \,|\, s_0, \pi\right] \geq 1 - \varepsilon \,, \tag{6.3}$$

where $\mathbb{P}[\cdot]$ denotes the probability operator, and $\varepsilon \in [0, 1]$ is a predefined *risk bound* indicating that the probability that the states stay within a safe set over the mission duration or planning horizon is at least $1 - \varepsilon$. We call the left hand term of (6.3), i.e., $\mathbb{P}\,[\, s_{1,\ldots,N} \in \mathbb{S} \,|\, s_0, \pi]$, Mission-Wide Probability of Safety (MWPS).

The resulting MWCC-OCP can be formally formulated as

$$J^*(s_0) = \min_{\pi} \; \mathbb{E}\left[\sum_{k=0}^{N-1} \ell(s_k, a_k) + \ell_N(s_N) \,\middle|\, s_0, \pi \right] \tag{6.4a}$$

$$\text{s.t.} \;\; \mathbb{P}\,[\, s_{1,\ldots,N} \in \mathbb{S} \,|\, s_0, \pi] \geq 1 - \varepsilon \,, \tag{6.4b}$$

for each feasible initial state $s_0 \in \mathbb{S}$. Here we assume that s_0 lies in the set from which there exists police sequence π such that (6.4b) is feasible, even through such set is not easy to be calculated [18]. We will call J^* the optimal cost function that assigns the optimal cost $J^*(s_0)$ to each feasible initial state s_0.

To solve problem (6.4), apart from the approximation methods mentioned in Section 6.1, one would naturally think of DP because it is arguably the most general approach to sequential decision-making problem when a model is at hand. Even when it is computationally expensive, DP may still serve as the basis for many practical approaches. Since the above mission-wide chance constraint are acting on the whole Markov Chain $s_{1,\ldots,N}$, existing methods for constrained DP [19] and constrained MDPs [20] cannot be used. This is due to the fact that these methods require that the state constraints are expressed in additive or independent form. As we will show in this chapter, problem (6.4) presents interesting features.

In the following sections, we integrate the constraint of problem (6.4) into its objective function via a penalty function, and then investigate conditions under which a DP scheme can be deployed.

6.3 MWPS-Constrained Problem with DP

Let $\mathbf{1}_{\mathbb{S}} : \mathcal{S} \to \{0,\, 1\}$ denote the indicator function of set $\mathbb{S} \subseteq \mathcal{S}$; $\mathbf{1}_{\mathbb{S}}(s) = 1$, if $s \in \mathbb{S}$, and 0 if $s \notin \mathbb{S}$. Let us define the set of functions $V_k^{\pi^k} : \mathbb{S} \to [0, 1]$, $\forall k = 0, 1 \ldots, N-1$ as follows:

$$V_N^{\pi^N}(s) = V_N(s) = \mathbf{1}_{\mathbb{S}}(s)\,,$$
$$V_k^{\pi^k}(s) = \mathbb{P}\left[\, s_{k+1,\ldots,N} \in \mathbb{S} \,\middle|\, s_k = s, \pi^k \right].$$

In this way, given a state s_0, the associated MWPS can be denoted by $V_0^{\pi}(s_0) = V_0^{\pi^0}(s_0) = \mathbb{P}\,[\, s_{1,\ldots,N} \in \mathbb{S} \,|\, s_0, \pi]\,.$

Lemma 6.1 *Function $V_k^{\pi^k}$ can be computed by the backward recursion:*

$$\begin{aligned} V_k^{\pi^k}(s_k) &= \int_{\mathbb{S}} V_{k+1}^{\pi^{k+1}}(s_{k+1}) \cdot \rho[s_{k+1} \mid s_k, \pi_k(s_k)] \, \mathrm{d}s_{k+1} \\ &:= \mathbb{E}_{s_{k+1}} \left[V_{k+1}^{\pi^{k+1}}(s_{k+1}) \middle| s_k, \pi_k(s_k) \right] \end{aligned} \tag{6.5}$$

initialized with the boundary condition $V_N(s_N) = \mathbf{1}_{\mathbb{S}}(s_N)$, where operation $\mathbb{E}_{s_{k+1}}$ indicates that the expectation is taken with respect to the probability distribution of s_{k+1} that remains in set $\mathbb{S}$.

Proof: See [21, Lemma 1] □

As what is typically done in constrained MDPs, penalizing the risk of mission failures with a suitably chosen cost function can in principle guarantee that the MDPs yield a policy which tends to not violate the constraints or even not violate the constraints at all when exact penalty is used.

Let us consider an optimization problem of the form:

$$\tilde{J}^*(s_0) = \min_{\pi} \; \mathbb{E} \left[\sum_{k=0}^{N-1} \ell(s_k, a_k) + \ell_N(s_N) \middle| s_0, \pi \right] + \zeta \left(V_0^{\pi}(s_0) \right), \tag{6.6}$$

where $\zeta : [0, 1] \to [-\infty, +\infty]$ is a penalty function mapping the MWPS into a scalar that possibly takes values in the extended real line.

Assumption 6.1 *The function ζ commutes with the expectation operator $\mathbb{E}_{s_{k+1}}$ for all $k = 0, 1, \ldots, N-1$, i.e.,*

$$\begin{aligned} \zeta \left(V_k^{\pi^k}(s_k) \right) &= \zeta \left(\mathbb{E}_{s_{k+1}} \left[V_{k+1}^{\pi^{k+1}}(s_{k+1}) \middle| s_k, \pi_k(s_k) \right] \right) \\ &= \mathbb{E}_{s_{k+1}} \left[\zeta \left(V_{k+1}^{\pi^{k+1}}(s_{k+1}) \right) \middle| s_k, \pi_k(s_k) \right] \end{aligned} \tag{6.7}$$

Remark 6.1 *It is obvious to verify that Assumption 6.1 holds if function ζ is affine and may only apply for this case. The commutation property in this assumption is essentially what we need to derive the following proposition.*

Proposition 6.1 *Let Assumption 6.1 be satisfied. Then* (6.6) *can be solved via the following DP recursion on the state-space:*

$$\begin{aligned}
&\tilde{J}_N(s_N) = \ell_N(s_N) + \zeta\left(V_N(s_N)\right) , \\
&\tilde{J}_k(s_k) = \min_{a_k \in \mathcal{A}} \ell(s_k, a_k) + \int_{\mathbb{S}} \tilde{J}_{k+1}(s_{k+1}) \rho[s_{k+1}|s_k, a_k] \mathrm{d}s_{k+1}, \quad k = N-1, \ldots, 0.
\end{aligned} \tag{6.8}$$

For every initial state s_0, the optimal cost $\tilde{J}^*(s_0)$ of problem (6.6) *is equal to $\tilde{J}_0(s_0)$, given by the last step (backward in time) of the above recursion. Furthermore, if $\pi_k^*(s_k) = a_k^*$ minimizes the right hand side of* (6.8) *for each s_k and k, then the policy $\pi^* = \{\pi_0^*, \ldots, \pi_{N-1}^*\}$ is optimal for problem* (6.6).

Proof: For $k = N-1, N-2, \ldots, 0$, let $\tilde{J}_k^*(s_k)$ be the optimal cost for the $(N-k)$-stage problem associated to (6.6) that starts at state s_k and time k, and ends at time N,

$$\tilde{J}_k^*(s_k) = \min_{\pi^k} \mathbb{E}\left[\sum_{i=k}^{N-1} \ell(s_i, a_i) + \ell_N(s_N) \middle| s_k, \pi^k \right] + \zeta\left(V_k^{\pi^k}(s_k) \right).$$

For $k = N$, we define $\tilde{J}_N^*(s_N) = \ell_N(s_N) + \zeta\left(V_N(s_N)\right)$. We will show by induction that the functions $\tilde{J}_k^*$ are equal to the functions $\tilde{J}_k$ generated by the DP recursion (6.8), such that for $k = 0$ the desired result will be obtained.

By definition, we have $\tilde{J}_N^* = \tilde{J}_N = \ell_N + \zeta$. Assume that for some $k+1$ and all s_{k+1}, we have $\tilde{J}_{k+1}^*(s_{k+1}) = \tilde{J}_{k+1}(s_{k+1})$. Then, since $\pi^k = (\pi_k, \pi^{k+1})$, we have for all $s_k \in \mathbb{S}$, (these developments are further explained hereafter)

$$\begin{aligned}
&\tilde{J}_k^*(s_k) = \min_{\pi^k} \mathbb{E}\left[\sum_{i=k}^{N-1} \ell(s_i, a_i) + \ell_N(s_N) \middle| s_k, \pi^k \right] + \zeta\left(V_k^{\pi^k}(s_k) \right) \\
&= \min_{\pi_k} \ell(s_k, \pi_k(s_k)) + \mathbb{E}_{s_{k+1}}\left[\min_{\pi^{k+1}} \mathbb{E}\left[\sum_{i=k+1}^{N-1} \ell(s_i, a_i) + \ell_N(s_N) \middle| s_{k+1}, \pi^{k+1} \right] \right] \\
&\qquad + \min_{\pi^{k+1}} \zeta\left(\mathbb{E}_{s_{k+1}}\left[V_{k+1}^{\pi^{k+1}}(s_{k+1}) \middle| s_k, \pi_k(s_k) \right] \right).
\end{aligned}$$

Based on equality (6.7) in Assumption 6.1, by commuting $\mathbb{E}_{s_{k+1}}$ and ζ, we can further observe that the above equation is equal to

$$\begin{aligned}\tilde{J}_k^*(s_k) = \min_{\pi_k} \ell(s_k, \pi_k(s_k)) \\ + \mathbb{E}_{s_{k+1}} \left[\min_{\pi^{k+1}} \mathbb{E} \left[\sum_{i=k+1}^{N-1} \ell(s_i, a_i) + \ell_N(s_N) \middle| s_{k+1}, \pi^{k+1} \right] \right] \\ + \min_{\pi^{k+1}} \mathbb{E}_{s_{k+1}} \left[\zeta \left(V_{k+1}^{\pi^{k+1}}(s_{k+1}) \right) \middle| s_k, \pi_k(s_k) \right]\end{aligned}$$

Furthermore, we observe that

$$\begin{aligned}\tilde{J}_k^*(s_k) &= \min_{\pi_k} \ \ell(s_k, \pi_k(s_k)) \\ &\quad + \mathbb{E}_{s_{k+1}} \left[\min_{\pi^{k+1}} \mathbb{E} \left[\sum_{i=k+1}^{N-1} \ell(s_i, a_i) + \ell_N(s_N) \middle| s_{k+1}, \pi^{k+1} \right] \right] \\ &\quad + \mathbb{E}_{s_{k+1}} \left[\min_{\pi^{k+1}} \zeta \left(V_{k+1}^{\pi^{k+1}}(s_{k+1}) \right) \middle| s_k, \pi_k(s_k) \right] \\ &= \min_{\pi_k} \ \ell(s_k, \pi_k(s_k)) + \mathbb{E}_{s_{k+1}} \left[\min_{\pi^{k+1}} \mathbb{E} \left[\sum_{i=k+1}^{N-1} \ell(s_i, a_i) + \ell_N(s_N) \middle| s_{k+1}, \pi^{k+1} \right] \right. \\ &\qquad \left. + \zeta \left(V_{k+1}^{\pi^{k+1}}(s_{k+1}) \right) \ \middle| \ s_k, \pi_k(s_k) \right].\end{aligned}$$

In the first equation above we moved the second $\min_{\pi^{k+1}}$ inside the brackets expression $\mathbb{E}_{s_{k+1}}[\,\cdot\,]$. In the last equation above, we integrated the two expressions of $\mathbb{E}_{s_{k+1}}[\,\cdot\,]$ into one. Finally, by using the definition of $\tilde{J}_{k+1}^*(s_{k+1})$, we can further have

$$\begin{aligned}\tilde{J}_k^*(s_k) &= \min_{\pi_k} \ \ell(s_k, \pi_k(s_k)) + \mathbb{E}_{s_{k+1}} \left[\tilde{J}_{k+1}^*(s_{k+1}) \middle| s_k, \pi_k(s_k) \right] \\ &= \min_{\pi_k} \ \ell(s_k, \pi_k(s_k)) + \mathbb{E}_{s_{k+1}} \left[\tilde{J}_{k+1}(s_{k+1}) \middle| s_k, \pi_k(s_k) \right] \\ &= \min_{a_k \in \mathcal{A}} \ \ell(s_k, a_k) + \int_{\mathbb{S}} \tilde{J}_{k+1}(s_k) \rho \left[s_{k+1} \mid s_k, a_k \right] \mathrm{d}s_{k+1} \\ &= \tilde{J}_k(s_k)\,.\end{aligned}$$

In the second equation, we used the induction hypothesis. In the third equation, we converted the minimization over π_k to a minimization over a_k, using the fact that for any function κ of s and a, we have

$$\min_{\mu \in \Pi} \kappa(s, \mu(s)) = \min_{a \in \mathcal{A}} \kappa(s, a)$$

where Π is the set of all functions $\mu(s)$ such that $\mu(s) \in \mathcal{A}$ for all s. $\square$

Remark 6.2 *If ζ is an affine function of the form $\zeta(x) = \lambda x + \Delta,\ \forall\, x \in [0,1]$, where $\lambda, \Delta \in \mathbb{R}$ are constants, it is easy to verify that this class of functions satisfy equality* (6.7) *in Assumption 6.1. Indeed, we observe that*

$$\begin{aligned}\zeta\left(V_k^{\pi^k}(s_k)\right) =& \zeta\left(\mathbb{E}_{s_{k+1}}\left[V_{k+1}^{\pi^{k+1}}(s_{k+1})\middle| s_k, \pi_k(s_k)\right]\right)\\ =&\lambda\mathbb{E}_{s_{k+1}}\left[V_{k+1}^{\pi^{k+1}}(s_{k+1})\middle| s_k, \pi_k(s_k)\right] + \Delta\\ =&\mathbb{E}_{s_{k+1}}\left[\lambda V_{k+1}^{\pi^{k+1}}(s_{k+1}) + \Delta\middle| s_k, \pi_k(s_k)\right]\\ =&\mathbb{E}_{s_{k+1}}\left[\zeta\left(V_{k+1}^{\pi^{k+1}}(s_{k+1})\right)\middle| s_k, \pi_k(s_k)\right],\end{aligned}$$

so that Proposition 6.1 holds; that is, the DP algorithm (6.8) *can be deployed to find an optimal solution for problem* (6.6).

Remark 6.3 *In the literature on stochastic reachability analysis, see e.g.[21], if the first term*

$$\mathbb{E}\left[\sum_{k=0}^{N-1} \ell(s_k, a_k) + \ell_N(s_N)\,\middle|\, s_0, \pi\right]$$

in the cost functions of (6.6) *is absent, a DP approach is proposed in [21] to minimize the risk only, that is,*

$$\min_{\pi}\ \zeta\left(V_0^{\pi}(s_0)\right) := 1 - V_0^{\pi}(s_0)\,,$$

which is a special case of problem (6.6). *Here, the parameters of ζ are given by $\lambda = -1$ and $\Delta = 1$. Moreover, any cost with a linear function on the probability of mission success as shown in Remark 6.2 falls in the category of problem* (6.6).

To further make problem (6.6) equivalent to the original problem (6.4), that is, $\tilde{J}^*(s_0) = \tilde{J}(s_0)$ for all feasible s_0, we need to specify a concrete function ζ.

Proposition 6.2 *Suppose that function ζ is of the form:*

$$\zeta(x) = \begin{cases} 0, & \textit{if } x \geq 1-\varepsilon \\ \infty, & \textit{otherwise}\,. \end{cases} \tag{6.9}$$

We observe that problem (6.4) *and problem* (6.6) *are equivalent.*

Proof: This proof is straightforward. Suppose that there is a set of policy sequences such that the MWCC (6.3) is satisfied. An optimal solution to problem (6.6) must fall in this set, because any policy sequence outside this set will yield an infinite penalty and hence cannot be optimal. Moreover, all the polices in this set result in zero penalty. Thus, the proof is concluded. □

However, Assumption 6.1 is not satisfied by function ζ defined in (6.9) because in this case ζ does not commute with the expectation operator $\mathbb{E}_{s_{k+1}}[\,\cdot\,]$. This means that in the case of function ζ given by (6.9), Proposition 6.1 is no longer applicable.

Note that the above observation implies that problem (6.6) with ζ defined in (6.9), which is equivalent to the original problem (6.4), needs to be further reformulated into the basic problem format where DP algorithm can be deployed. This will be the subject of the following section.

6.4 Functional State Augmentation

The observations of the previous section show that adding penalty function associated to the MWPS into the objective function is of limited use if that penalty is to be nonlinear. Indeed, DP solutions only exist for the cases where the is an affine function of the MWPS. For example, if the exact penalty function (6.9) is required, the DP scheme presented in Proposition 6.1 fails to apply.

These observations, however, do not exclude the existence of a solution to problem (6.4) via DP, but they exclude the existence of classical cost-to-go functions expressed in terms of the state s_k alone. We now discuss how one can in principle deal with situations where Assumption 6.1 is violated and strict MWCC requirement should be satisfied. A solution via DP arguably requires one to augment the state space to enlarge the information at time k involved in the decision-making.

To that end, let us define the sequence of functions $F_k : \mathcal{S} \to [0, \infty)$, for $k = 0, 1 \ldots, N$ as follows:

$$
\begin{aligned}
F_0(s) &= \mathbf{1}_{\mathbb{S}}(s)\,,\\
F_k(s) &= \mathbb{P}\left[\, s_{1,\ldots,k-1} \in \mathbb{S} \wedge s_k = s \,|\, s_0, \pi \,\right],
\end{aligned}
$$

This sequence has the forward linear dynamics:

$$
F_{k+1}(s) = \int_{\mathbb{S}} F_k(s')\rho\left[s_{k+1} = s \,|\, s_k = s', a_k\right] \mathrm{d}s'. \tag{6.10}
$$

Given a state s_0, the associated MWPS then reads as

$$\int_{\mathbb{S}} F_N(s)\mathrm{d}s = \mathbb{P}\left[\, s_{1,\dots,N} \in \mathbb{S} \,|\, s_0, \pi \right].$$

At this point, by integrating the MWCC into the cost function and using Proposition 6.2, we can then reformulate the MWCC-OCP problem (6.4) into its equivalent form

$$J^*(s_0) = \min_{\pi} \; \mathbb{E}\left[\sum_{k=0}^{N-1} \ell(s_k, a_k) + \ell_N(s_N) \,\middle|\, s_0, \pi \right] + \zeta\left(\int_{\mathbb{S}} F_N(s)\mathrm{d}s \right), \tag{6.11}$$

where the function ζ is given by e.g. (6.9).

It is useful to observe that s_k is a vector in $\mathbb{R}^{n_s}$ stochastically decided by s_k and a_k through (6.1), while F_k is a functional in some functional space deterministically decided by given F_0 and policy $\pi_0, \dots, \pi_{k-1}$ through (6.10). Let us denote the augmented state as $\delta_k = (s_k, F_k)$, consisting of a vector state and a functional state. The dynamics of state s_k and F_k are given by (6.1) and (6.10), respectively. Naturally, the control a_k should now depend on the new state δ_k, or equivalently, a policy sequence should consist of policies π_k based on the regular state s_k, as well as the functional state F_k.

By using the new augmented state, (6.11) is identical to the basic problem format of finite-horizon optimal control problem, but includes a terminal constraint on the additional state F_N. Consequently, we can retain a DP solution to (6.11) without the need of Assumption 6.1, at the expense of a significantly more complex state-space to manipulate. The following Proposition details that DP solution.

Proposition 6.3 . *For every feasible initial state s_0, the optimal cost $J^*(s_0)$ of problem* (6.11) *is equal to $J_0(\delta_0)$, given by the last step of the following backward-iteration algorithm:*

$$\begin{aligned} J_N(\delta_N) &= \ell_N(s_N) + \zeta\left(\int_{\mathbb{S}} F_N(s)\mathrm{d}s \right), \\ J_k(\delta_k) &= \min_{a_k} \; \ell(s_k, a_k) + \int_{\mathbb{S}} J_{k+1}(\delta_{k+1})\rho[s_{k+1}|s_k, a_k]\mathrm{d}s_{k+1}, \; k = N-1, \dots, 0. \end{aligned} \tag{6.12}$$

Furthermore, if $\pi_k^*(\delta_k) = a_k^*$ minimizes the right hand side of (6.12) *for each $\delta_k = (s_k, F_k)$ and k, then the policy $\pi^* = \{\pi_0^*, \dots, \pi_{N-1}^*\}$ is optimal for problem* (6.11).

Proof: For $k = N-1, N-2, \ldots, 0$, let $J_k^*(\delta_k)$ be the optimal cost for the $(N-k)$-stage problem that starts at state $\delta_k = (s_k, F_k)$ and time k, and ends at time N, that is,

$$J_k^*(\delta_k) = \min_{\pi^k} \mathbb{E}\left[\sum_{i=k}^{N-1} \ell(s_i, a_i) + \ell_N(s_N) \,\middle|\, s_k, \pi^k\right]$$

For $k = N$, we define $J_N^*(\delta_N) = \ell_N(s_N) + \zeta\left(\int_{\mathbb{S}} F_N(s)\mathrm{d}s\right)$. We will show by induction that the functions J_k^* are equal to the functions J_k generated by the DP algorithm (6.12), such that at $k = 0$ the desired result will be obtained.

By definition, we have $J_N^* = J_N = \ell_N + \zeta$. Assume that for some $k+1$ and all δ_{k+1}, we have $J_{k+1}^*(\delta_{k+1}) = J_{k+1}(\delta_{k+1})$. Then, since $\pi^k = (\pi_k, \pi^{k+1})$, we have for all δ_k

$$\begin{aligned}
J_k^*(\delta_k) &= \min_{\pi^k} \mathbb{E}\left[\sum_{i=k}^{N-1} \ell(s_i, a_i) + \ell_N(s_N) \,\middle|\, s_k, \pi^k\right] \\
&= \min_{\pi_k} \ell(s_k, \pi_k(\delta_k)) + \mathbb{E}_{s_{k+1}}\left[\min_{\pi^{k+1}} \mathbb{E}\left[\sum_{i=k+1}^{N-1} \ell(s_i, a_i) + \ell_N(s_N) \,\middle|\, s_{k+1}, \pi^{k+1}\right]\right]
\end{aligned}$$

Finally, by using the definition of $J_{k+1}^*(\delta_{k+1})$, we can further have

$$\begin{aligned}
J_k^*(\delta_k) &= \min_{\pi_k} \ell(s_k, \pi_k(\delta_k)) + \mathbb{E}_{s_{k+1}}\left[J_{k+1}^*(\delta_{k+1})\right] \\
&= \min_{\pi_k} \ell(s_k, \pi_k(\delta_k)) + \mathbb{E}_{s_{k+1}}\left[J_{k+1}(\delta_{k+1})\right] \\
&= \min_{a_k \in \mathcal{A}} \ell(s_k, a_k) + \int_{\mathbb{S}} J_{k+1}(\delta_{k+1}) \rho\left[s_{k+1} \mid s_k, a_k\right] \mathrm{d}s_{k+1} \\
&= J_k(\delta_k)\,,
\end{aligned}$$

The explanations to the equations above are analogous to the corresponding ones given in the proof of Proposition 6.1. Moreover, since $F_0(s_0) \equiv 1$ for every feasible initial state $s_0 \in \mathbb{S}$, we have $J^*(s_0) = J_0(\delta)$. $\square$

Remark 6.4 *The DP principle given by Proposition 6.3 actually holds for any penalty function ζ, i.e., not just for the case of* (6.9)*, which makes Proposition 6.3 more universal. However, in order to enforce the exact MWCC, then ζ defined in* (6.9) *can be used.*

As is typically the case, state augmentation often comes at a price of making very complex state and/or control spaces. This state augmentation has a complex state space that comprise a regular Euclidean space and a functional space. Nonetheless, the exact dynamic programming scheme proposed in Proposition 6.3, for the first time, gives the exact global optimal solution for the MWCC-OCP. Therefore, Proposition 6.3 can serve as a stepping-stone for possibly many approximate dynamic programming methods to be developed in the future.

6.5 Numerical Illustration

In this section, we analytically demonstrate how to use the DP algorithm proposed in the preceding section on a simple example. Consider the one-dimensional system of the form

$$s_{k+1} = s_k + a_k + w_k, \qquad k = 0, 1,$$

with initial state $s_0 = \bar{s}$ which is supposed to be feasible regarding the following problem setup. Here, $a_k \in [-0.1, 0.1]$ and disturbance $w_k \sim \mathcal{N}(0, 0.0001)$. The safe set $\mathbb{S} = [-1, 1]$. The mission duration $N = 2$. The cost functions $\ell(s, a) = s^2 + a^2$ and $\ell_2(s) = s^2$. The exact penalty function ζ follows from (6.9) with risk bound $\varepsilon = 0.1$. We observe that the augmented functional states $F_1,\ F_2,\ F_3$ are given by:

$$\begin{aligned}
F_0(s_0) &= \mathbf{1}_{[-1,1]}(\bar{s}) = 1\,,\\
F_1(s) &= F_0(\bar{s})\rho\,[s_1 = s \,|\, s_0 = \bar{s}, a_0 = x_0\,]\\
&= \frac{1}{0.01\sqrt{2\pi}} e^{-\frac{(s-\bar{s}-x_0)^2}{0.002}}\\
F_2(s) &= \int_{-1}^{1} F_1(s')\rho\left[s_2 = s \,\middle|\, s_1 = s', a_1 = x_1\,\right] \mathrm{d}s'\\
&= \int_{-1}^{1} \frac{1}{(0.01\sqrt{2\pi})^2} e^{-\frac{(s'-\bar{s}-x_0)^2+(s-s'-x_1)^2}{0.002}} \mathrm{d}s'
\end{aligned}$$

Note that the functions $F_1,\ F_2$ are functions involving parameters $x_0,\ x_1 \in [-0.1, 0.1]$.

By using the augmented state $\delta_k = (s_k, F_k)$, the DP algorithm takes the following form. At stage 2, we initialize the value function

$$\begin{aligned}
J_2(\delta_2) = J_2(s_2, F_2) &= \ell_N(s_2) + \zeta\left(\int_{-1}^{1} F_2(s)\mathrm{d}s\right)\\
&= \begin{cases} s_2^2, & \text{if } \zeta\left(\int_{-1}^{1} F_2(s)\mathrm{d}s\right) \geq 0.9\\ \infty, & \text{otherwise}\,. \end{cases}
\end{aligned}$$

for all possible state δ_2, At stage 1, we solve the following optimization problem for all possible δ_1

$$\begin{aligned} J_1(\delta_1) &= J_1(s_1, F_1) \\ &= \min_{a_1 \in [-0.1, 0.1]} \ell(s_1, a_1) + \mathbb{E}_{s_2}\left[J_2(\delta_2) \mid s_1, a_1\right] \\ &= \min_{a_1 \in [-0.1, 0.1]} s_1^2 + a_1^2 + \int_{-1}^{1} \frac{1}{0.01\sqrt{2\pi}} e^{-\frac{(s_2 - s_1 - a_1)^2}{0.002}} \cdot J_2(\delta_2) \mathrm{d}s_2 \,, \end{aligned}$$

for all possible state δ_1. The optimal control policy π_1 is given by solving the above optimization problem.

At stage 0, for the fixed initial state $s_0 = \bar{s}$, the optimal control input $\pi_0(s_0)$ is given by solving the following optimization problem

$$\begin{aligned} J^*(s_0) &= J_0(\delta_0) = J_0(\bar{s}, F_0) \\ &= \min_{a_0 \in [-0.1, 0.1]} \ell(s_0, a_0) + \mathbb{E}_{s_1}\left[J_1(\delta_1) \mid s_0 = \bar{s}, a_0\right] \\ &= \min_{a_0 \in [-0.1, 0.1]} a_0^2 + \int_{-1}^{1} J_1(\delta_1) \frac{1}{0.01\sqrt{2\pi}} e^{-\frac{(s_1 - \bar{s} - a_0)^2}{0.002}} \mathrm{d}s_1 \end{aligned}$$

This "toy" example illustrates the essential procedures of using Proposition 6.3. A practical implementation on a more complete example is beyond the scope of this chapter and is being investigated in our current work.

6.6 Concluding Remarks

In this chapter, we investigate solutions for mission-wide chance-constrained optimal control problems via Dynamic Programming. We show that classic Dynamic Programming recursions on the state-space of the problem are possible if the penalty imposed on mission-wide chance constraints violations commutes with the expected value operator underlying the stochastic dynamics. We show that this requirement is not fulfilled when imposing hard mission-wide chance constraints. We then present a state augmentation that tackles the problem. The resulting augmented space consists of a regular Euclidean space and a functional space. The proposed dynamic programming scheme for the mission-wide chance-constrained optimal control problems can hopefully play a fundamental role for developing approximation methods, because it characterizes the optimal solutions.

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Chapter 7

Stochastic MPC with Mission-Wide Chance Constraints

This Chapter is based on the following paper:

Kai Wang and Sébastien Gros. Recursive Feasibility of Stochastic Model Predictive Control with Mission-Wide Probabilistic Constraints. In *Proceedings of the 60th IEEE Conference on Decision and Control (CDC)*, 2021.

7.1 Introduction

Model predictive control (MPC) has been well established for dealing with complex constrained optimal control problems [1], [2]. In the context of MPC, the system dynamics are required to be known and deterministic. In practice, the system uncertainty, including imprecise model parameters and process noises, is generally unavoidable. Because MPC does not take the uncertainty into account, constraints violations can occur.

For applications where safety is critical, robust MPC strategies have been proposed to explicitly account for uncertainty. However, robust MPC can only handle bounded disturbances and the resulting control strategy can be conservative. To overcome these limitations, stochastic MPC methods have been developed to seek a trade-off between control performance and the risk of constraint violations using chance constraints.

There are mainly three forms of chance constraints proposed in the literature: individual chance constraint, stage-wise chance constraint [3] and mission-wide chance constraint [4]. E.g. in path planning for vehicles in the presence of obstacles, individual or stage-wise constraints restrict at every time instant the probability that the vehicle collides with an obstacle. In contrast, a mission-wide chance constraint directly restricts the probability of collision on the overall driving mission. A mission-wide chance constraint is arguably more meaningful than stage-wise constraints. Indeed, the former directly handles the risk of running a mission [4], [5], while the latter does it very indirectly. However, stage-wise chance constraints are easier to handle than mission-wide constraints. Indeed, the latter handles probabilities over entire state trajectories, yielding very large probability spaces. More forms of chance constraints are discussed in, e.g., [3, Section 2.2] and references therein.

The current research on stochastic MPC focuses on developing efficient methods for solving the underlying optimization problem, while recursive feasibility is less explored. Indeed, because of the (possibly unbounded) stochasticity, the recursive feasibility of stochastic MPC typically holds in the probabilistic sense, making its analysis much more involved. Some results exist in specific contexts. When the system uncertainties have bounded supports, recursive feasibility can be guaranteed using robust MPC [6], at the cost of yielding very conservative control policies. For linear stochastic systems with unbounded support, if the first two moments of the disturbance distribution are known, constraint-tightening methods via the Chebyshev–Cantelli inequality are presented in [7]–[9]. Recursive feasibility is guaranteed using backup strategies when an infeasible optimization problem is encountered [7], [8], and using time-varying risk bound [9]. The author in [10], [11] proposed stochastic MPC algorithms that have a certain probability of remaining feasible if the initial condition is feasible. However, none of these methods tackle mission-wide probability of safety (MWPS), nor can provide a meaningful certificate of MWPS. In [12], the problem of maximizing the MWPS is expressed as a stochastic invariance problem and further developed into an optimal control problem, which is solvable via dynamic programming. Unfortunately, problems constraining the MWPS rather than maximizing it cannot necessarily be put in that simple form.

Guaranteeing recursive feasibility of a stochastic MPC problem with MWPS constraints is an open problem, and this chapter investigates a tentative solution. The main contributions of this chapter is threefold. First, we show that if a policy is designed to achieve a certain MWPS, then the MWPS remaining until the end of the mission remains constant in the expected value sense. Second, we design a recursively feasible control scheme using shrinking horizon policies in the context of stochastic MPC with

MWPS guarantee. The proposed scheme treats directly the probability of running a mission successfully and therefore does not introduce artificial conservativeness. Third, an efficient scenario-based algorithm is proposed to deploy the idea in the linear case.

The rest of this chapter is structured as follows. In Section 7.2 we present the problem statement of stochastic MPC with MWPS constraints, and its difference from the classical stochastic MPC with stage-wise probabilistic constraint. Section 7.3 details how the MWPS remains constant throughout the mission, and a recursively feasible policy design is discussed in Section 7.4. We demonstrate the idea in the linear stochastic MPC case based on an efficient scenario-based algorithm in Section 7.5. Section 7.6 provides a numerical case study to illustrate the proposed method. Finally, Section 7.7 concludes the chapter points to some future work.

Notations: The notation $\mathbb{P}[\,\cdot\,|\,\cdot\,]$ denotes a conditional probability. The notation $\mathbb{E}_{\mathfrak{B}}[\,\cdot\,]$ indicates expectation conditional on event $\mathfrak{B}$. We use $s_{0,\ldots,N} \in \mathbb{S}$ to denote that a state sequence $s_{0,\ldots,N}$ lies in a constraint set $\mathbb{S}$ of the state space, i.e., $s_k \in \mathbb{S}$ for all $k = 0, \ldots, N$. We denote $\mathbb{I}_{[a,b]}$ the set of integers in the interval $[a, b] \subseteq \mathbb{R}$.

7.2 Problem Statement

7.2.1 MWPS Constrained Optimal Control

We consider a mission spanning a predefined horizon $N \in \mathbb{N}$ to be "safe" if:

$$s_{1,\ldots,N} \in \mathbb{S} \tag{7.1}$$

starting from some initial states $s_0 \in \mathbb{S}$. Here, $s_k \in \mathbb{R}^n$ is the state at time step k, and $\mathbb{S} \subset \mathbb{R}^n$ is a set in the state space. We assume that the true stochastic system dynamics are given by:

$$\rho[\,s_+ \,|\, s, a\,] \tag{7.2}$$

providing the probability density underlying transitions from a state-input pair s, a to a new state s_+. Throughout the chapter, we assume that the states are known and continuous. Notice that the control community typically uses:

$$s_+ = f(s, a, w) \tag{7.3}$$

to describe stochastic dynamics, in which w denotes the stochastic disturbances and f is generally a nonlinear function. The input a is given by a control policy sequence

$$\forall k \in \mathbb{I}_{[0,N-1]}, \quad \pi := \{\pi_0, \ldots, \pi_{N-1}\}$$

such that

$$\forall k \in \mathbb{I}_{[0,N-1]} \quad a_k = \pi_k(s_k).$$

In general, guaranteeing the absolute safety as described in (7.1) yields very conservative control policies, or is even infeasible if the uncertainty is unbounded. Alternatively, for a given initial condition $s_0 \in \mathbb{S}$ and a policy sequence π, we are interested in the Mission-Wide Probability of Safety (MWPS):

$$\mathbb{P}[s_{1,\dots,N} \in \mathbb{S} \,|\, s_0, \pi]. \tag{7.4}$$

The problem we are interested in is then to find a policy sequence solution of

$$\min_{\pi} \quad \mathbb{E}\left[M(s_N) + \sum_{k=0}^{N-1} L(s_k, \pi_k(s_k))\right] \tag{7.5a}$$

$$\text{s.t.} \quad \mathbb{P}[s_{1,\dots,N} \in \mathbb{S} \,|\, s_0, \pi] \geq S\,, \tag{7.5b}$$

where $S \in [0, 1]$ is a predefined safety bound, the functions L and M are some given stage and terminal costs, and (7.5a) is the expectation over the state trajectories resulting from s_0, π and (7.2).

In practice, calculating an optimal policy sequence for problem (7.5) exactly is hardly possible, because it involves optimization over an infinite dimensional function space. To tackle this issue, we will be interested in using stochastic MPC formulations to generate policies that enforce the MWPS (7.4), where a control input a_k is computed by solving an optimal control problem at every time step. In that context, a key concept will be the remaining MWPS at any time $k \in \mathbb{I}_{[1,N-1]}$ for a given state s_k, defined as:

$$\mathbb{P}[s_{k+1,\dots,N} \in \mathbb{S} \,|\, s_k, \pi] \tag{7.6}$$

Here s_k is the outcome of a realization $s_{1,\dots,k}$ of the Markov Chain and its relationship to (7.4).

A key observation is that a policy sequence π satisfying (7.5b) does not yield any guarantee on (7.6). Indeed, an adversarial realization can e.g. bring the system into a state s_k for which the remaining MWPS is lower than S.

This observation entails that in the proposed context, the notion of recursive feasibility needs to be treated in a different way that is commonly done in robust MPC. We will detail this in Section 7.3. We briefly detail next the motivation for developing methods to treat MWPS constraints outside of the classical chance-constraint framework.

7.2.2 Mission-wide Constraints & Stage-Wise Constraints

In this section, we present our motivation for treating MWPS directly rather than via Stage-Wise Probability of Safety (SWPS). In particular, regardless of the desired MWPS level, enforcing it via SWPS becomes very conservative for long missions. SWPS problems seek policies that enforce constraints of the form:

$$\mathbb{P}\left[s_k \in \mathbb{S} \,|\, s_0,\, \pi\right] \geq s_k \geq s, \quad \forall\, k \in \mathbb{I}_{[1,N]}\,, \tag{7.7}$$

in which $1 \geq s_k \geq s \geq 0$. One can then easily verify that the Boolean algebra and Booles's inequality entail that:

$$\begin{aligned}
\mathbb{P}[s_{1,\dots,N} \in \mathbb{S} \,|\, s_0, \pi] &= 1 - \mathbb{P}\left[\bigcup_{k=1}^{N} s_k \notin \mathbb{S} \,\middle|\, s_0, \pi\right] \\
&\geq 1 - \sum_{k=1}^{N} \mathbb{P}\left[s_k \notin \mathbb{S} \,|\, s_0, \pi\right] \\
&= 1 - \sum_{k=1}^{N} \left(1 - \mathbb{P}\left[s_k \in \mathbb{S} \,|\, s_0, \pi\right]\right) \\
&\geq 1 - N + \sum_{k=1}^{N} s_k \geq 1 - N + Ns\,.
\end{aligned}$$

To ensure the satisfaction of (7.5b) via imposing (7.7), requires the choice, $1-N+Ns \geq S$, i.e. a bound for s can be derived as:

$$s \geq \frac{N-1}{N} + \frac{S}{N}\,. \tag{7.8}$$

Hence enforcing MWPS (7.4) via SWPS (7.7) requires selecting s according to (7.8), which yields a bound s close to one very fast as N increases, see Fig. 7.1, hence turning the SWPS constraints into hard constraints. While tighter bounds than (7.8) can be derived[1], treating MWPS via SWPS without introducing conservativeness is difficult. The intuitive reason behind this issue is that SWPS formulations neglect the time-correlation between the constraints violations, and as a result, it offers an incorrect representation of the risks incurred by a system over a mission. Bound (7.8) corrects that, at the cost of introducing a high conservatism. By using a similar argument to what we developed above, risk-allocation technology proposed in [13] optimizes the

[1] e.g. the Bonferroni inequalities allow one to refine the bound in (7.8) by accounting for some of the correlation between successive states

risk assigned to each stage-wise constraint instead of using constant risk in (7.7). This method leads a computationally expensive two-stage optimization problem and is still conservative as depicted in [3, Fig. 1].

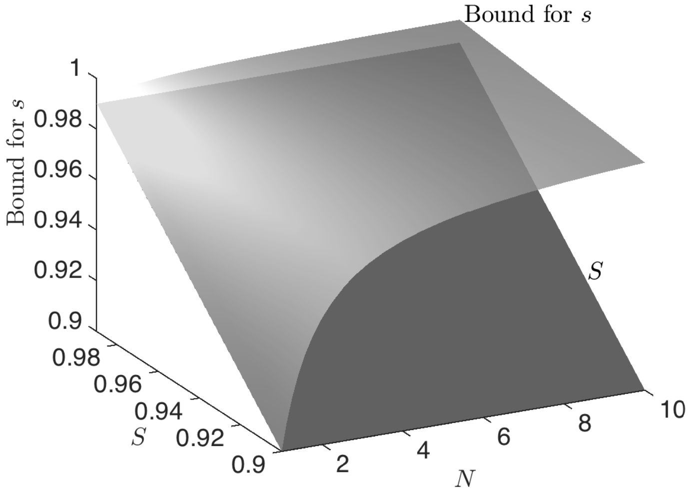

Figure 7.1: Illustration of bound (7.8) for various N and S. The curved manifold displays the bound (7.8) for the SWPS such that a prescribed MWPS (7.4) holds.

7.3 Relation between Remaining and Initial MWPS

Here, we show that the remaining MWPS is constant in the expected value sense. This offers a novel path for guaranteeing the recursive feasibility of MPC-like control schemes with MWPS constraints.

Lemma 7.1 *If the policy sequence π satisfies* (7.4)*, then*

$$\mathbb{E}_{\{s_{1,\dots,k}\in\mathbb{S}\,|\,s_0,\pi\}}\left[\mathbb{P}[s_{k+1,\dots,N}\in\mathbb{S}\,|\,s_k,\pi]\right]\geq S \tag{7.9}$$

for all $k=1,\dots,N-1$, i.e. the remaining MWPS at time k satisfies the constraint on the prescribed MWPS (7.4) *in the expected value sense.*

Proof: We observe that:

$$\mathbb{P}[s_{1,\dots,N} \in \mathbb{S} \,|\, s_0, \pi] = \int_{\mathbb{S}^k} \mathbb{P}[s_{1,\dots,k}|s_0, \pi]\mathbb{P}[s_{k+1,\dots,N} \in \mathbb{S}\,|s_{0,\dots,k}, \pi]\mathrm{d}s_1 \dots \mathrm{d}s_k$$
$$= \int_{\mathbb{S}^k} \mathbb{P}[s_{1,\dots,k} \,|\, s_0, \pi]\mathbb{P}[s_{k+1,\dots,N} \in \mathbb{S} \,|\, s_k, \pi]\, \mathrm{d}s_1 \dots \mathrm{d}s_k$$
$$= \int_{\mathbb{S}} \mathbb{P}[s_{1,\dots,k-1} \in \mathbb{S} \wedge s_k|s_0, \pi]\, \mathbb{P}[s_{k+1,\dots,N} \in \mathbb{S}\,|s_k, \pi]\mathrm{d}s_k$$
$$:= \mathbb{E}_{\{s_{1,\dots,k} \in \mathbb{S} \,|\, s_0, \pi\}} \left[\mathbb{P}[s_{k+1,\dots,N} \in \mathbb{S} \,|\, s_k, \pi] \right] .$$

Here, $s_{1,\dots,k}$ is a Markov Chain underlying (7.2), and therefore a random variable in the high dimensional space $(\mathbb{R}^n)^k$. Given a policy sequence π, the remaining MWPS at time k, i.e., the term inside $\mathbb{E}_{\{s_{1,\dots,k} \in \mathbb{S} \,|\, s_0, \pi\}}[\,\cdot\,]$, depends only on the random state s_k, which is a particular dimension in the Markov Chain $s_{1,\dots,k}$. Hence, the last equation holds because here $\mathbb{E}_{\{s_{1,\dots,k} \in \mathbb{S} \,|\, s_0, \pi\}}[\,\cdot\,]$ is used to denote the expectation value of the remaining MWPS that is taken over all possible realizations of the random Markov Chain $s_{1,\dots,k}$ that remains in $\mathbb{S}$. □

Lemma 7.1 entails that the MWPS is conserved in the expected value sense throughout the mission if a mission-wide policy sequence has been selected at the beginning of the mission. Result (7.9) is arguably best interpreted in a frequentist framework. Indeed, even though a specific realization $s_{0,\dots,k}$ may be adversarial for the remaining MWPS, we observe that in average the MWPS remains unchanged throughout the mission. As a result, (7.9) entails that if running missions under policy π designed according to (7.5), the resulting ratio of success will asymptotically be at least S. While this statement may appear tautological, it provides a basic concept of recursive feasibility that can be translated into constraints in a MPC framework to ensure that a prescribed MWPS is achieved. We detail this observation below.

7.4 Recursive Feasibility of MWPS with Shrinking-Horizon Policies

In this section, we focus on solving the originally proposed mission-wide probability-constrained finite-horizon optimal control problem (7.5) using shrinking-horizon policies that are updated as the mission progresses. The reason behind this is that the exact optimal policies for (7.5) is difficult to compute in general.

As a result, in practice, the policy sequence π is typically finitely parameterized, and hence restricted to a subset of the set of admissible policies. This introduces sub-optimality, and makes it useful to re-solve problem (7.5) at every time instant k, according to the latest state realization s_k. We then consider at every time k the control policy sequence:

$$\pi^k = \left\{\pi_k^k, \ldots, \pi_{N-1}^k\right\} \tag{7.10}$$

lasting to the end of the mission. For the sake of brevity, we will work with a shrinking horizon extending to the end of the mission. The fixed, receding horizon shorter than the mission duration will be the object of our future work.

At every time instant $k \in \mathbb{I}_{[0,N-1]}$, for the corresponding state s_k, we consider solving the following shrinking-horizon, mission-wide and chance-constrained problem:

$$\min_{\pi^k} \quad \mathbb{E}\left[M(s_N) + \sum_{l=k}^{N-1} L(s_l, \pi_l^k(s_l))\right] \tag{7.11a}$$

$$\text{s.t.} \quad \mathbb{P}[s_{k+1,\ldots,N} \in \mathbb{S} \,|\, s_k, \pi^k] \geq S_k \tag{7.11b}$$

to get a new policy sequence. Here, $S_k \in [0, 1]$ is a varying risk-bound that will be specified later. Notice that while (7.11) computes an entire policy sequence π^k for the current state s_k, only the first policy π_k^k of that sequence is used to generate the actual control action, as the policy sequence is recalculated at the next time instant $k + 1$, in a classic MPC fashion. The inputs eventually applied to the closed-loop system will therefore read as:

$$a_k = \pi_k^k\left(s_k\right), \quad \forall\, k \in \mathbb{I}_{[0,N-1]} \tag{7.12}$$

In the context of mission-wide stochastic MPC, we will consider the recursive feasibility issue of employing the policy sequence $\{\pi_0^0, \ldots, \pi_{N-1}^{N-1}\}$ resulting from extracting only the first policy π_k^k of the policy sequence π^k at every time step k, for all $k \in \mathbb{I}_{[0,N-1]}$. We show next that retaining recursive feasibility in the sense of (7.9) requires only that the new policy sequence produces a remaining MWPS that is not worse than a discounted one achieved by the previous policy for the current state s_k. We formalise this statement in the proposition below.

Proposition 7.1 *Assume that the initial policy sequence π^0 satisfies the MWPS cosntraint:*

$$\mathbb{P}[s_{1,\ldots,N} \in \mathbb{S} \,|\, s_0, \pi^0] \geq S_0 \geq S \tag{7.13}$$

and that each policy sequence π^k is built under the constraint:

$$\mathbb{P}[s_{k+1,\ldots,N} \in \mathbb{S} \,|\, s_k, \pi^k] \geq S_k \tag{7.14}$$

where

$$S_k = \gamma_k \mathbb{P}[s_{k+1,\ldots,N} \in \mathbb{S} \,|\, s_k, \pi^{k-1}]$$

holds and with $\gamma_k \in (0,1]$*, for all* $k \in \mathbb{I}_{[1,N-1]}$*. Then the MWPS under* $a_k = \pi_k^k(s_k)$ *and* $k \in \mathbb{I}_{[0,N-1]}$ *reads as:*

$$\mathbb{P}\left[s_{1,\ldots,N} \in \mathbb{S} \,\middle|\, s_0, \{\pi_0^0, \ldots, \pi_{N-1}^{N-1}\}\right] \geq \prod_{k=1}^{N-1} \gamma_k S_0 \,. \tag{7.15}$$

Proof: We will prove this by induction. Consider

$$\begin{aligned}
&\mathbb{P}\left[s_{1,\ldots,N} \in \mathbb{S} \,\middle|\, s_0, \{\pi_0^0, \ldots, \pi_k^k, \ldots, \pi_{N-1}^k\}\right] \\
=&\int_{\mathbb{S}} \mathbb{P}\left[s_{1,\ldots,k-1} \in \mathbb{S} \wedge s_k \,|\, s_0, \{\pi_0^0, \ldots, \pi_{k-1}^{k-1}\}\right] \mathbb{P}\left[s_{k+1,\ldots,N} \in \mathbb{S} \,\middle|\, s_k, \pi^k\right] \mathrm{d}s_k \\
\geq &\int_{\mathbb{S}} \mathbb{P}\left[s_{1,\ldots,k-1} \in \mathbb{S} \wedge s_k \,|\, s_0, \{\pi_0^0, \ldots, \pi_{k-1}^{k-1}\}\right] \gamma_k \mathbb{P}\left[s_{k+1,\ldots,N} \in \mathbb{S} \,\middle|\, s_k, \pi^{k-1}\right] \mathrm{d}s_k \\
=&\gamma_k \mathbb{P}\left[s_{1,\ldots,N} \in \mathbb{S} \,\middle|\, s_0, \{\pi_0^0, \ldots, \pi_{k-1}^{k-1}, \ldots, \pi_{N-1}^{k-1}\}\right] ,
\end{aligned}$$

where the last equality holds because the last integral describes the MWPS associated to applying the policy sequence $\{\pi_0^0, \ldots, \pi_{k-1}^{k-1}, \ldots, \pi_{N-1}^{k-1}\}$. Hence an induction from

$$\mathbb{P}[s_{1,\ldots,N} \in \mathbb{S} \,|\, s_0, \pi^0] \geq S_0$$

yields (7.15). □

Let us introduce the following corollaries, showing the practical implications of Proposition 7.1:

Corollary 7.1 ***(Guarantee of MWPS)*** *The choice:*

$$\prod_{k=1}^{N-1} \gamma_k S_0 = S \tag{7.16}$$

together with the policy update constraint (7.14) *yields a sequence of policies* $\{\pi_0^0, \ldots, \pi_{N-1}^{N-1}\}$ *that satisfies the prescribed MWPS constraint* (7.13).

Proof: The update constraint (7.14) ensures that (7.15) holds. Condition (7.16) imposed on the factors $\gamma_{1,\ldots,N-1}$ then ensures that (7.5b) is satisfied. □

Corollary 7.2 ***(Recursive Feasibility)*** *Constraint* (7.14) *is always feasible for any* $\gamma \leq 1$.

Proof: We observe that (7.14) is feasible for $\pi^k = \pi^{k-1}$. $\square$

7.5 A Scenario-Based Stochastic MPC Approach

7.5.1 Simplified Control Policy for Stochastic MPC

In this section we deploy the mission-wide stochastic MPC idea developed so far in the linear case. Let us consider that the stochastic dynamics (7.3) are explicitly given by:

$$s_{k+1} = As_k + Ba_k + w_k \,, \tag{7.17}$$

and that the safe set $\mathbb{S}$ is polytopic, i.e.

$$\mathbb{S} = \{\, s \,|\, Cs + c \leq 0\} \,. \tag{7.18}$$

Here we assume that the disturbances w_k, $k \in \mathbb{I}_{[0,N-1]}$ are i.i.d., and zero-mean for the sake of notation convenience.

At each time instants $k \in \mathbb{I}_{[0,N-1]}$, the predicted state s_t for all $t = k, k+1, \ldots, N$, can be split into a nominal part and an stochastic error part, i.e., $s_t = \bar{s}_t + e_t$. we consider the policy sequence π_t^k parameterized via $\bar{a}_t,\ K$, given by:

$$a_t = \pi_t^k\left(s_t\right) := \bar{a}_t + Ke_t, \quad \forall t \in \mathbb{I}_{[k,N-1]}$$

where K is a stabilizing feedback matrix for the nominal dynamics:

$$\bar{s}_{t+1} = A\bar{s}_t + B\bar{a}_t \quad \bar{s}_k = s_k. \tag{7.19}$$

The stochastic error dynamics are then given by:

$$e_{t+1} = \left(A + BK\right)e_t + w_t, \quad e_k = 0\,. \tag{7.20}$$

Our goal is to solve the following mission-wide probability constrained optimal control problem at every time instant k:

$$\min_{\bar{a}_{k,\ldots,N-1}} \ \mathbb{E}\left[s_N^\top Q_N s_N + \sum_{t=k}^{N-1}\left(s_t^\top Q s_t + a_t^\top R a_t\right)\right] \tag{7.21a}$$

$$\text{s.t.} \quad \bar{s}_k = s_k \tag{7.21b}$$

$$\bar{s}_{t+1} = A\bar{s}_t + B\bar{a}_t, \qquad \forall t \in \mathbb{I}_{[k,N-1]} \tag{7.21c}$$

$$e_{t+1} = \left(A + BK\right)e_t + w_t, \ \forall t \in \mathbb{I}_{[k,N-1]} \tag{7.21d}$$

$$s_{t+1} = \bar{s}_{t+1} + e_{t+1}, \qquad \forall t \in \mathbb{I}_{[k,N-1]} \tag{7.21e}$$

$$\mathbb{P}[Cs_{t+1} + c \leq 0, \ \forall t \in \mathbb{I}_{[k,N-1]}] \geq S_k\,. \tag{7.21f}$$

Here, Q, Q_N are semi-positive definite, R is positive definite, and the value

$$S_k = \gamma_k \mathbb{P}[s_{k+1,\dots,N} \in \mathbb{S} \mid s_k, \pi^{k-1}] \tag{7.22}$$

will be estimated at every time instant k using Monte Carlo simulation based on the real, closed-loop state s_k and the previous policy sequence π^{k-1}.

7.5.2 An Efficient Scenario-Based Stochastic MPC Algorithm

Cost Function. Since $\mathbb{E}[s_t] = \bar{s}_t$ and e_t is zero mean (since w_t is zero mean by assumption), the cost function (7.21a) can be written explicitly as

$$\bar{s}_N^\top Q_N \bar{s}_N + \sum_{t=k}^{N-1} \left(\bar{s}_t^\top Q \bar{s}_t + \bar{a}_t^\top R \bar{a}_t \right) + \sigma,$$

where σ is a constant term that can be excluded from the cost function.

Chance Constraint. Substituting (7.21d) and (7.21e) into (7.21f), the constraints can be rewritten as

$$\mathbb{P}[C(\bar{s}_{t+1} + (A + BK)\, e_t + w_t) + c \leq 0, \forall t \in \mathbb{I}_{[k,N-1]}] \geq S_k$$

and further be written as

$$\mathbb{P}[\underbrace{\mathcal{C}\mathcal{A}w_{k,\dots,N-1}^\top + [c, \dots, c]^\top}_{:=H} + \mathcal{C}\bar{s}_{k+1,\dots,N}^\top \leq 0] \geq S_k \tag{7.23}$$

with matrix $\mathcal{A}$ obtained by condensing the dynamic (7.21d) and matrix $\mathcal{C}$ being block-diagonal with C as blocks.

Scenario Approximation. In general, providing a closed form for (7.23) is difficult. Fortunately, this problem can be handled efficiently with a scenario-based approach. Constraints (7.23) is replaced by a finite, sufficiently large number N_k of deterministic constraints resulting from sampling the disturbance sequence $w_{k,\dots,N-1}$. For a given time instant k, we define the i^{th} sample for all $i \in \mathbb{I}_{[1,N_k]}$ as

$$w_{k,\dots,N-1}^{(i)} := \{w_k^{(i)}, \dots, w_{N-1}^{(i)}\},$$

Hence, the chance constraint (7.23) can be converted to

$$H^{(i)} + \mathcal{C}\bar{s}_{k+1,\dots,N}^\top \leq 0, \quad \forall i \in \mathbb{I}_{[1,N_k]}, \tag{7.24}$$

where

$$H^{(i)} = \mathcal{C}\mathcal{A}(w^{(i)}_{k,\dots,N-1})^\top + [c,\dots,c]^\top\,.$$

In order to guarantee that (7.24) approximates (7.23) with a high probability $1-\beta$, where β is typically set to be very small (e.g., $\beta = 10^{-6}$), N_k must satisfy the following inequality [14]:

$$\sum_{n=1}^{d_k}\binom{N_k}{n}(1-S_k)^n S_k^{N_k-n} \le \beta\,,$$

where d_k is the number of optimization variables. The explicit lower bound of N_k can be further derived as [15]:

$$N_k \ge \frac{2}{1-S_k}\left(\ln\frac{1}{\beta}+d_k\right). \tag{7.25}$$

To further reduce the conservatism of the scenario-based approach, a sample removal approach is proposed in [16] and several variants are proposed. Their use here is beyond the scope of this chapter.

For each scenario i, n_c linear constraints are generated in (7.24). It is clear that n_c is equal to the number of rows of matrix $H^{(i)}$. We additionally observe that for the constraint of index $j \in \mathbb{I}_{[1,n_\mathrm{c}]}$ in (7.24), the following inequality holds:

$$[H^{(i)}]_j + [\mathcal{C}]_j\bar{s}_{k+1,\dots,N} \le \max_{q\in\mathbb{I}_{[1,N_k]}}[H^{(q)}]_j + [\mathcal{C}]_j\bar{s}_{k+1,\dots,N}$$

for all $i \in \mathbb{I}_{[1,N_k]}$, where $[\star]_j$ denotes the j^{th} row of the matrix $\star$. Note that this inequality is tight, i.e., for all constraint of index j there always exists at least one sample of index i that ensures the above inequality tight. Hence j^{th} constraint is satisfied for all realizations i if they are satisfied for the one having the largest $[H^{(i)}]_j$.

Let us label:

$$\mathcal{I}_j = \max_{i\in\mathbb{I}_{[1,N_k]}}[H^{(i)}]_j, \quad \forall j \in \mathbb{I}_{[1,n_\mathrm{c}]}.$$

then we have that constraint (7.24) is equivalent to the following constraints:

$$\mathcal{I}_j + [\mathcal{C}]_j\bar{s}_{k+1,\dots,N} \le 0, \quad \forall j \in \mathbb{I}_{[1,n_\mathrm{c}]}.$$

Note that calculating $\mathcal{I}_j$, for all $j \in \mathbb{I}_{[1,n_\mathrm{c}]}$, requires only n_c (vector) maximum operations that are easy to implement and computationally efficient.

Now, (7.21) is equivalent to the following QP:

$$\min_{\bar{a}_{k,\dots,N-1}} \quad \bar{s}_N^{\mathrm{T}} Q_N \bar{s}_N + \sum_{t=k}^{N-1} \left(\bar{s}_t^{\mathrm{T}} Q \bar{s}_t + \bar{a}_t^{\mathrm{T}} R \bar{a}_t \right) \tag{7.26a}$$

$$\text{s.t.} \quad \bar{s}_k = s_k \tag{7.26b}$$

$$\bar{s}_{t+1} = A\bar{s}_t + B\bar{a}_t + \bar{w}_t, \ \forall t \in \mathbb{I}_{[k,N-1]} \tag{7.26c}$$

$$\mathcal{I}_j + [\mathcal{C}]_j \bar{s}_{k+1,\dots,N} \leq 0, \quad \forall j \in \mathbb{I}_{[1,n_c]} \tag{7.26d}$$

yielding a regular QP of the same complexity as a normal linear MPC.

A systematic overview of the proposed scenario-based mission-wide linear stochastic MPC scheme is summarized in Algorithm 2.

Algorithm 2 linear stochastic MPC with MWPS constraints

Initialization: S_0, $\gamma_{1,\dots,N-1}$, initial state s_0

For $k = 0 : N-1$, **Do**

1)**If** $k \geq 1$ Evaluate S_k in (7.22) through Monte Carlo simulation

2) Generate N_k scenarios according to (7.25)

3) Get the solution $\bar{a}^*_{k,\dots,N-1}$ by solving (7.26)

4) Send $\bar{a}^*_k$ to the actual system and update state: $s_{k+1} = As_k + B\bar{a}^*_k + w_k$

7.6 Numerical Illustration

We consider the linear system (7.17) with

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

and the uncertainty is assumed to have a Gauss distribution

$$w_k \sim \mathcal{N}(0, 0.04 \cdot I).$$

The safe (constraint) set $\mathbb{S}$ (7.18) is given by matrices

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad c = \begin{bmatrix} -2 \\ -2 \\ -10 \\ -2 \end{bmatrix}.$$

The matrices $Q = I$, $R = 0.1$, and

$$K = [-0.6167, -1.2703]\,, \quad Q_N = \begin{bmatrix} 2.0599 & 0.5916 \\ 0.5916 & 1.4228 \end{bmatrix}$$

are computed from the corresponding LQR solution.

We select $N = 11$, $S_0 = 0.98$ and $\gamma_{1,\ldots,10} = 0.99$, resulting in $S = \prod_{k=1}^{10} \gamma_k S_0 = 0.8863$. The number N_k of disturbance sample is selected from (7.25). The bound S_k given by (7.22) is evaluated from Monte Carlo simulation and $\beta = 10^{-6}$. In the simulations, we observed that $S_k \approx 0.99$ for all $k \in \mathbb{I}_{[1,10]}$. This is due to N_k calculated from (7.25) is conservative, such that the remaining MWPS at time k achieved by the previous policy sequence $\{\pi_k^{k-1}, \ldots, \pi_{N-1}^{k-1}\}$, is much higher than that is actually required.

A Monte Carlo simulation that simulates 10^5 missions shows that the resulting ratio of mission success is 99.88%. This result is larger than $S = 88.63\%$. The reason for this discrepancy is that the scenario-based method adopted is conservative. Figure 7.2 shows the state trajectories of 10^3 missions.

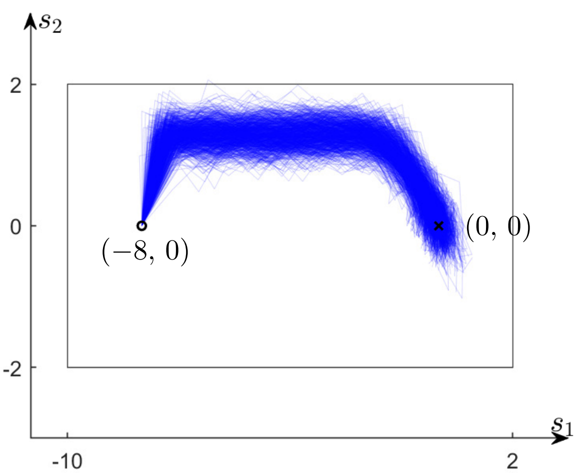

Figure 7.2: State trajectories plot obtained by running 10^3 number of missions starting from the initial sate $s_0 = [-8,\ 0]^\top$. The reference point is $[0,\ 0]^\top$. The rectangular area depicts the safe set $\mathbb{S}$.

7.7 Concluding Remarks

We investigated optimal policies satisfying Mission-Wide Probability of Safety constraints, i.e. constraints imposing the safety of a system over an entire mission. This is in contrast with classical stochastic MPC, where safety constraints are imposed independently at every time stage. We show that recursive feasibility holds in the expected value sense for the concept of Mission-Wide Probability of Safety, opening a simple and practically meaningful concept of recursive feasibility for stochastic MPC. Optimal control with mission-wide probabilistic constraints is challenging. However, a computationally efficient scenario-based approach is proposed to solve this issue for linear stochastic problems. For the sake of brevity, a shrinking-horizon approach was presented in this chapter. The scenario-based approach proposed here relies on classical Monte-Carlo sampling. More advanced methods will be developed in the future for the proposed method.

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Chapter 8

Conclusions and Future Work

8.1 Conclusions

This thesis presents a collection of contributions in the field of model-based predictive control with a focus on the explicit consideration of uncertainty. For Chapters 3-7, comprehensive conclusions are presented at the end of each chapter. Three primary contributions emerge from this work.

1. We have introduced a refined tube-based robust model predictive control approach that is simple but yet computationally efficient in Chapter 3. This method not only revisits the three fundamental formulations but also 1) simplifies the offline computations by employing support functions and 2) improves the desirable control theoretic properties compared to its predecessors.

2. We have delved into stochastic model predictive control problems with stage-wise chance constraints in Chapters 4 and 5. The method proposed in Chapter 4 is able to reject possibly unbounded stochastic disturbances by utilizing an offline computed probabilistic positively invariant set, while the method proposed in Chapter 5 leads to exponentially convergence of the closed-loop system by utilizing probabilistic reachable sets and with the use of the initialization strategy similar to that of Chapter 3. The proposed two stochastic model predictive control methods both maintain robust recursive feasibility and stability, with standard quadratic programming online and simple linear programming offline.

3. We have extended the literature on stage-wise chance constraints with the introduction of mission-wide chance constraints to guarantee safety in stochastic

optimal control scenarios. To tackle these challenges, we have introduced two approaches: dynamic programming for exact problem resolution and stochastic model predictive control with a shrinking horizon for approximate solutions.

8.2 Future Research Directions

Certainly, there exists a wide spectrum of future research directions worthy of exploration. Firstly, the methodologies of model-based predictive control under uncertainty presented in this thesis hold significant potential for real-world applications. While we have only illustrated the proposed algorithms on numerical examples for testing purposes in this thesis, it is important to note that uncertainty is a key factor in nearly all engineering processes. Therefore, there exist numerous intriguing applications of these methods yet to be explored. Additionally, the methodologies introduced in this thesis also pave the way for theoretical progress. Several significant and promising direction for further investigation are outlined below:

- **Asymptotic Stability Guarantees for Stochastic Model Predictive Control:** Despite advancements in stochastic model predictive control over the past two decades, there remains a lack of a systematic treatment of stabilizing conditions, particularly when considering expectation cost functions, chance constraints, and unbounded stochastic uncertainty simultaneously. Therefore, delving into this open yet interesting problem would be worthwhile.

- **Distributionally Robust Setting:** Given that it is often the case that only partial knowledge about the uncertainty distribution is available, operating within the distributionally robust framework is more desirable. While there is some literature emerged more recently in this area, the computational overhead is generally much higher compared to robust and stochastic model predictive control. The challenge primarily stems from the complexities involved in propagating the set of distributions. Existing techniques often facing exponentially increasing numbers of decision variables and constraints in the underlying optimization problems. Therefore, there is a necessity to develop more computationally efficient control algorithms to enable their application in real-case scenarios.

- **Output-Feedback Setting:** The majority of model predictive control approaches in the presence of uncertainty are designed for scenarios with full state feedback. However, in numerous practical applications, states cannot be fully measured.

Some robust and stochastic model predictive controllers have been developed within the output-feedback setting, but stability and/or convergence guarantees are often lacking, except for a few tube-based robust model predictive control approaches. With the development of state observers and disturbance observers, delving deeper into this problem will be promising and practically demanded.

- **Adaptive (Dual), Data-Driven and Learning-Based Control:** Given the presence of inevitable uncertainty, it is desirable to enhance model predictive control with persistent input excitation or collected historical data for adapting or learning model parameters in the face of uncertainty. Naturally, this also raises several theoretical challenges, such as ensuring safety certificates and analyzing closed-loop stability properties.

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