Tube MPC with Time-Varying Cross-Sections

Kai Wang, Sixing Zhang*, Sébastien Gros and Saša V. Raković

***Abstract*—This article considers tube model predictive control of discrete-time linear systems subject to additive bounded disturbances and mixed state and control constraints. An improved tube model predictive controller, leveraging the advantages and mitigating the disadvantages of three pivotal existing methods, is proposed. Its computational aspects and theoretical properties are thoroughly discussed and compared with its predecessors. Two numerical examples are provided to illustrate the benefits of the proposed method.**

***Index Terms*—model predictive control; robust control; constrained systems; tube model predictive control.**

I. Introduction

Model Predictive Control (MPC) is a widely used, modern, optimization-based control technique [1], [2]. Robust MPC is an improved form of MPC, which explicitly models the influence of uncertainty and aims to maintain stability, performance and constraint satisfaction [3]. The range of robust MPC proposals includes open-loop minimax robust MPC [4], [5], theoretically complete and computationally intensive closed-loop robust MPC [6], [7], robust MPC based on dynamic programming [2], [8], disturbance affine feedback MPC [9], [10] and tube MPC [11]–[20] offering theoretically flexible and computationally tractable methods.

As pointed out in the review articles [3], [21], tube MPC has emerged as a leading paradigm for robust MPC due to its intuitive ease, computational simplicity and guaranteed control-theoretic properties. The robust MPC proposals [11], [12] were reported almost simultaneously, and their formulations can be integrated into the framework of tube MPC [15]. A sequence of articles appeared over the last two decades, which has defined the state-of-the-art of tube MPC. The developed tube MPC methods include the so-called rigid tube MPC [13], homothetic tube MPC [16], elastic tube MPC [19] and parameterized tube MPC [18] for linear systems. Tube MPC extensions for nonlinear systems, output feedback systems and data-driven systems, and their applications have also been widely investigated [2], [22]. More recently, the rigid tube MPC using implicit representations of terminal set and tube cross-section sets was reported in [14], which enables its utility for higher dimensional systems.

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The primary focus of this article is to refine the early and simple, but yet computationally effective, tube MPC methods [11]–[13] for constrained discrete-time linear systems with additive bounded disturbances. The tube MPC proposals [11]–[13] are effectively concerned with the same problem, and they possess both computational simplicity and strong control-theoretic properties. These methods are computationally efficient since they reduce to conventional MPC applied to deterministic nominal systems subject to modified stage constraints, appropriate terminal constraints, and disturbance-free stage and terminal cost functions. These methods also provide guarantees of robust convergence/stability and robust positive invariance. At the conceptual level, all these methods employ rather simple tube parameterizations and affine control policies in constructing the tube optimal control problems. On the other hand, proposals [11]–[13] differ in terms of local uncertainty propagation, nominal state initialization and modified constraints as well as utilized cost functions. Naturally, these differences lead to different features of these proposals, and each proposal has its own advantages and disadvantages, as discussed in more details in Section V-A.

Contributions: In this article, we revisit the early tube MPC methods [11]–[13] and report a refinement of these methods. The mixed state and control constraints are considered, which are more general than the separate state and control constraints specified in [11]–[13]. A novel state decomposition strategy, that further decomposes the local uncertainty into two components, is proposed. Based on this new decomposition strategy, we propose a tube MPC method that improves both robust convergence of [11] and robust asymptotic stability of [12] to robust exponential stability, and it enlarges the effective domains of [12], [13] to a robust positively invariant set that is identical with the effective domain of [11] by construction. Moreover, adapting recent ideas of [14], the support function is employed in constraint tightening to avoid the computationally expensive, explicit construction of tube cross-section sets. Due to the employment of support functions, the disturbance set, which contains the origin in its interior, need not be polytopic, and it can be merely convex and compact.

Paper Structure: Section II presents the problem setup and article objectives. Section III describes the constraints on the employed tubes and specifies the set of admissible decision variables, and it gives the utilized cost functions. Section IV proposes the improved tube MPC and analyzes its control-theoretic properties. Section V provides a detailed comparison and offers two numerical illustrations, and it ends with concluding remarks. Proofs of technical results are given in Appendix.

Basic Nomenclature and Conventions: The sets of real numbers, non-negative integers and positive integers are denoted by $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{N}_+$, respectively. Given $a, b \in \mathbb{N}$, with $a < b$, we use the notation $\mathbb{N}_{[a:b]}$ to denote the set of non-negative integers $\{a, a+1, \ldots, b\}$. We denote $\mathbb{N}_{[0:b]}$ by $\mathbb{N}_b$. Given $p \in [1, \infty]$, the p–norm of a vector $x \in \mathbb{R}^n$ is denoted by $\|x\|_p$, while $\mathcal{B}_p^n$ denotes the corresponding closed unit p–norm ball in $\mathbb{R}^n$. For convenience, we denote the Euclidean norm $\|\cdot\|_2$ by $\|\cdot\|$. Given a symmetric and positive definite matrix $M \in \mathbb{R}^{n\times n}$ and a vector $x \in \mathbb{R}^n$, we denote $x^\top M x$ by $\|x\|_M^2$. The Minkowski sum of two nonempty sets $\mathcal{X}$ and $\mathcal{Y}$ in $\mathbb{R}^n$ is given by $\mathcal{X} \oplus \mathcal{Y} := \{x + y \,:\, x \in \mathcal{X},\ y \in \mathcal{Y}\}$. The image of a nonempty set $\mathcal{X}$ under a matrix of compatible dimensions M is given by $M\mathcal{X} := \{Mx \,:\, x \in \mathcal{X}\}$. Likewise, if M is a square matrix, for any integer $k \in \mathbb{N}$, $M^k\mathcal{X} := \{M^k x \,:\, x \in \mathcal{X}\}$. A proper D–set in $\mathbb{R}^n$ is a closed convex subset of $\mathbb{R}^n$ that contains the origin in its interior. A proper C–set in $\mathbb{R}^n$ is a bounded proper D–set in $\mathbb{R}^n$. The intersection of finitely many closed half–spaces is a polyhedral set. A polytopic set is a bounded polyhedral set. The support function $\mathrm{h}(\mathcal{X}, \cdot)$ of a nonempty, closed, convex set $\mathcal{X} \subseteq \mathbb{R}^n$ is given, for all $y \in \mathbb{R}^n$, by

$$\mathrm{h}(\mathcal{X}, y) := \sup_x \{y^\top x \,:\, x \in \mathcal{X}\}.$$

Finally, we do not distinguish row vectors from column vectors unless necessary, as no confusion should arise.

II. Preliminaries

A. System and Constraints

We consider discrete-time linear, time-invariant, uncertain systems given by

$$x^+ = Ax + Bu + w, \tag{2.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $w \in \mathbb{R}^n$ are, respectively, the current state, control and disturbance, and $x^+ \in \mathbb{R}^n$ denotes the successor state. The matrix pair $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ is *a priori* given. At any current time, the state x is known, while the disturbance w is unknown but obeys the constraint

$$w \in \mathcal{W}. \tag{2.2}$$

Throughout this article, we consider mixed polyhedral state and control constraints

$$\begin{aligned}&(x, u) \in \mathcal{Y} \text{ with}\\ &\mathcal{Y} := \left\{(x, u) \,:\, \forall i \in \mathcal{I}_\mathcal{Y},\ c_i^\top x + d_i^\top u \le 1\right\}.\end{aligned} \tag{2.3}$$

We make the following standing assumptions on the system (2.1) and constraints (2.2)–(2.3).

Assumption 1:

- The matrix pair $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ is known, and it is strictly stabilizable.
- The disturbance constraint set $\mathcal{W} \subseteq \mathbb{R}^n$ is a proper C–set, and its support function $\mathrm{h}(\mathcal{W}, \cdot)$ is point-wise computable.
- The index set $\mathcal{I}_\mathcal{Y} := \{1, 2, \ldots, n_\mathcal{Y}\}$, with $n_\mathcal{Y} \in \mathbb{N}_+$, is finite, and for all $i \in \mathcal{I}_\mathcal{Y}$, $(c_i, d_i) \in \mathbb{R}^{n+m}$ are known. The set $\mathcal{Y}$ is a polyhedral proper D–set, and its representation is irreducible.

B. State Decomposition

In this article, the parameterized state and control predictions and an affine control policy are employed. For a state $x \in \mathbb{R}^n$, a control policy $\Pi_{N-1}(\cdot)$ is a sequence of affine control laws $\{\pi_k(\cdot,\cdot,\cdot,\cdot)\}_{k\in\mathbb{N}_{N-1}}$, each term of which is parameterized via points $x_k \in \mathbb{R}^n$, $z_k \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^m$ and an *a priori* specified control matrix $K \in \mathbb{R}^{m\times n}$ as follows

$$\forall k \in \mathbb{N}_{N-1},\ \pi_k(x_k, z_k, v_k, x) := v_k + K(x_k - z_k), \tag{2.4}$$

so that

$$\forall k \in \mathbb{N}_{N-1},\ x_{k+1} = Ax_k + B(v_k + K(x_k - z_k)) + w_k, \tag{2.5}$$

starting from $x_0 = x$.

Assumption 2: The matrix $K \in \mathbb{R}^{m\times n}$ is known, and it is such that the matrix $A_K := A + BK$ is strictly stable.

The sequences $\{z_k\}_{k\in\mathbb{N}_N}$ and $\{v_k\}_{k\in\mathbb{N}_{N-1}}$ satisfy the following nominal dynamics

$$\forall k \in \mathbb{N}_{N-1},\ z_{k+1} = Az_k + Bv_k. \tag{2.6}$$

The predicted states are decomposed into three components

$$\forall k \in \mathbb{N}_N,\ x_k = z_k + \bar{s}_k + \tilde{s}_k. \tag{2.7}$$

The nominal component z_k follows from the nominal controlled dynamics (2.6), and the local uncertain components $\bar{s}_k$ and $\tilde{s}_k$ are specified by the following two dynamics

$$\forall k \in \mathbb{N}_{N-1},\ \bar{s}_{k+1} = A_K\bar{s}_k \text{ with } \bar{s}_0 = x - z_0, \text{ and} \tag{2.8}$$

$$\forall k \in \mathbb{N}_{N-1},\ \tilde{s}_{k+1} = A_K\tilde{s}_k + w_k \text{ with } \tilde{s}_0 = 0. \tag{2.9}$$

The initial local uncertainty $\bar{s}_0$ represents the deviation of the current state x from the initial nominal state z_0. Note that the component $\bar{s}_k$ evolves through the *deterministic* dynamics (2.8) starting from $\bar{s}_0$, while the component $\tilde{s}_k$ evolves through the *uncertain* dynamics (2.9) starting from the origin. A moment of reflection reveals that dynamics (2.5) is the superposition of dynamics (2.6), (2.8) and (2.9).

C. Tube Parameterization

The effect of the uncertainty $\tilde{s}_k$ is accounted for by employing the set-dynamics

$$\forall k \in \mathbb{N}_{N-1},\ \widetilde{S}_{k+1} = A_K\widetilde{S}_k \oplus \mathcal{W} \text{ with } \widetilde{S}_0 = \{0\}, \tag{2.10}$$

induced by the local uncertainty dynamics (2.9) and the disturbance set $\mathcal{W}$. It follows from (2.10) that

$$\forall k \in \mathbb{N}_{[1,N]},\ \widetilde{S}_k = \bigoplus_{i=0}^{k-1} A_K^i\mathcal{W} \text{ and } \widetilde{S}_0 = \{0\}. \tag{2.11}$$

The state tube is a sequence of sets, denoted by $\mathbf{X}_N := \{X_k\}_{k\in\mathbb{N}_N}$, in which each set $X_k \subseteq \mathbb{R}^n$ is parameterized via $z_k \in \mathbb{R}^n$, $\bar{s}_k \in \mathbb{R}^n$ and the set $\widetilde{S}_k \subseteq \mathbb{R}^n$ as follows

$$\forall k \in \mathbb{N}_N,\ X_k := z_k + \bar{s}_k \oplus \widetilde{S}_k. \tag{2.12}$$

For all $k \in \mathbb{N}_N$, $z_k + \bar{s}_k$ and $\widetilde{S}_k$ are, respectively, the centers and cross-sections of the tube $\mathbf{X}_N$.

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D. Article Objectives

As discussed in the review article [23], exploring the differences of the competing methods, such as [11] and [12], is necessary for properly assessing as well as advancing existing approaches to robust and stochastic MPC. In this article, we aim to introduce an improved tube MPC that combines best principal components of [11], [12], and the rigid tube MPC [13]. The improved tube MPC deploys the prediction tubes $\mathbf{X}_N$ with time-varying tube cross-section sets $\widetilde{S}_k$, $k \in \mathbb{N}_N$, and it makes use of the nominal state initialization strategy analogously to [13] and the tube terminal constraints compatibly with [11] as well as the cost function considered in [12], [13].

III. Tube Constraints, Decision Variables and Cost Function

A. Initialization

This article adopts the initialization strategy as follows

$$\bar{s}_0 = x - z_0 \text{ and } \bar{s}_0 \in \mathcal{X}_f. \tag{3.1}$$

Assumption 3: The set $\mathcal{X}_f \subseteq \mathbb{R}^n$ is the maximal robust positively invariant set for the system $x^+ = A_K x + w$, and constraints $(x, Kx) \in \mathcal{Y}$ and $w \in \mathcal{W}$.

The maximal robust positively invariant set $\mathcal{X}_f$ is the limit of the following standard set iteration [24]

$$\forall j \in \mathbb{N},\ \mathcal{X}_{j+1} := \{x \ : \ A_K x \oplus \mathcal{W} \subseteq \mathcal{X}_j\} \bigcap \mathcal{X}_0 \text{ with}$$
$$\mathcal{X}_0 := \{x \ : \ (x, Kx) \in \mathcal{Y}\},$$

i.e., $\mathcal{X}_f := \mathcal{X}_\infty$. It is known that the maximal robust positively invariant set [24] is nonempty if and only if the minimal robust positively invariant set $\mathcal{X}_\mathcal{O}$ for $x^+ = A_K x + w$ with $w \in \mathcal{W}$, specified by

$$\mathcal{X}_\mathcal{O} := \bigoplus_{j=0}^{\infty} A_K^j \mathcal{W}, \tag{3.2}$$

is a subset of $\mathcal{X}_0$. The polyhedral structure of the limit set $\mathcal{X}_\infty$, however, is not necessarily guaranteed when the limit is not finitely determined. It is also well known that the set $\mathcal{X}_\infty$ is finitely determined when one of the iterates $\mathcal{X}_j$ is bounded and $\mathcal{X}_\mathcal{O} \subseteq \text{interior}(\mathcal{X}_0)$ [24]. More generally, the maximal robust positively invariant set is finitely determined when $\mathcal{X}_j \subseteq \mathcal{X}_{j+1}$ for some $j \in \mathbb{N}$, in which case $\mathcal{X}_\infty = \mathcal{X}_j$ is finitely determined and it is nonempty when, in addition, $\mathcal{X}_\mathcal{O} \subseteq \mathcal{X}_0$.

Throughout this article, we assume that $\mathcal{X}_f$ is finitely determined and nonempty (thus, $\mathcal{X}_f$ is at least a polyhedral proper D–set in $\mathbb{R}^n$) and $\mathcal{X}_\mathcal{O}$ is strictly admissible with respect to the stage constraints (i.e., that, for all $x \in \mathcal{X}_\mathcal{O}$, we have $(x, Kx) \in \text{interior}(\mathcal{Y})$). These natural conditions are summarized by the following assumption.

Assumption 4:

- The set $\mathcal{X}_f \subseteq \mathbb{R}^n$ is a polyhedral proper D–set, and its irreducible representation is given, for known vectors $r_i \in \mathbb{R}^n$, $i \in \mathcal{I}_{\mathcal{X}_f} := \{1, 2, \ldots, n_f\}$ with $n_f \in \mathbb{N}_+$, by
$$\mathcal{X}_f := \{x \ : \ \forall i \in \mathcal{I}_{\mathcal{X}_f},\ r_i^\top x \leq 1\}. \tag{3.3}$$
- The set $\mathcal{X}_\mathcal{O}$ is such that $\mathcal{X}_\mathcal{O} \subseteq \text{interior}(\mathcal{X}_0)$, or, equivalently its support function satisfies
$$\forall i \in \mathcal{I}_\mathcal{Y},\ \text{h}\left(\mathcal{X}_\mathcal{O}, c_i + K^\top d_i\right) < 1. \tag{3.4}$$

B. Stage Constraints

Based on the sets X_k in (2.12) and the affine control laws $\pi_k(\cdot,\cdot,\cdot,\cdot)$ in (2.4), the tube stage constraints take the form

$$\forall k \in \mathbb{N}_{N-1},\ (z_k, \bar{s}_k, v_k) \in \mathcal{G}_k \text{ with}$$
$$\mathcal{G}_k := \left\{(z, \bar{s}, v) : \forall \tilde{s} \in \widetilde{S}_k,\ (z + \bar{s} + \tilde{s}, v + K\bar{s} + K\tilde{s}) \in \mathcal{Y}\right\}. \tag{3.5}$$

To handle these constraints in a computationally practicable manner, we make use of the support functions of sets $\widetilde{S}_k$ given in (2.11), which allow us to avoid the explicit implementation of the underlying set-algebraic operations. By applying [14, Proposition 1], the structural properties of sets $\mathcal{G}_k$, $k \in \mathbb{N}_{N-1}$ can be summarized by the following proposition.

Proposition 1: Suppose Assumptions 1, 2 and 4 hold. For all $k \in \mathbb{N}_{N-1}$, the stage constraint sets $\mathcal{G}_k$ are polyhedral proper D–sets in $\mathbb{R}^{2n+m}$ with (possibly redundant) representations

$$\mathcal{G}_k := \left\{(z, \bar{s}, v) : \forall i \in \mathcal{I}_\mathcal{Y},\ c_i^\top z + \eta_i^\top \bar{s} + d_i^\top v \leq 1 - f_{(k,i)}\right\},$$

where, for all $i \in \mathcal{I}_\mathcal{Y}$, with $\eta_i := c_i + K^\top d_i$, the scalars $f_{(k,i)} \in [0, 1)$ are specified by

$$f_{(k,i)} := \text{h}\left(\widetilde{S}_k, \eta_i\right) = \sum_{j=0}^{k-1} \text{h}\left(\mathcal{W},\ (A_K^j)^\top \eta_i\right). \tag{3.6}$$

C. Terminal Constraints

The tube terminal constraint takes the form

$$(z_N, \bar{s}_N) \in \mathcal{H}_f \text{ with}$$
$$\mathcal{H}_f := \left\{(z, \bar{s}) : \forall \tilde{s} \in \widetilde{S}_N,\ z + \bar{s} + \tilde{s} \in \mathcal{X}_f\right\}. \tag{3.7}$$

Analogously to the construction of sets $\mathcal{G}_k$ in Proposition 1, the structural properties of the set $\mathcal{H}_f$ can be summarized as per the following.

Proposition 2: Suppose Assumptions 1–4 hold. The set $\mathcal{H}_f$ is a polyhedral proper D–set in $\mathbb{R}^{2n}$, with a (possibly redundant) representation

$$\mathcal{H}_f := \{(z, \bar{s}) \ : \ \forall i \in \mathcal{I}_{\mathcal{X}_f},\ r_i^\top z + r_i^\top \bar{s} \leq 1 - g_i\},$$

where, for all $i \in \mathcal{I}_{\mathcal{X}_f}$, the scalars $g_i \in [0, 1)$ are specified by

$$g_i := \text{h}\left(\widetilde{S}_N, r_i\right) = \sum_{j=0}^{N-1} \text{h}\left(\mathcal{W},\ (A_K^j)^\top r_i\right). \tag{3.8}$$

In the following remark, we discuss the computational aspects of the relevant scalars $f_{(k,i)}$ and g_i.

Remark 1: As shown in (3.6) and (3.8), the evaluation of the underlying scalars $f_{(k,i)}$ and g_i requires merely the evaluation of a number of support functions of $\mathcal{W}$. Moreover, these scalars can be computed by the following simple iterations given, with $f_{(0,i)} = 0$ and $g_{(0,i)} = 0$, by

$$\forall k \in \mathbb{N}_{N-2}, \forall i \in \mathcal{I}_\mathcal{Y},\ f_{(k+1,i)} = f_{(k,i)} + \text{h}\left(\mathcal{W}, (A_K^k)^\top \eta_i\right),$$
$$\forall j \in \mathbb{N}_{N-1}, \forall i \in \mathcal{I}_{\mathcal{X}_f},\ g_{(j+1,i)} = g_{(j,i)} + \text{h}\left(\mathcal{W}, (A_K^j)^\top r_i\right),$$

in which we set $g_i := g_{(N,i)}$. The evaluation of support functions of frequently encountered convex and compact sets can be found in the beginning of [25, Appendix]. A more detailed discussion of the evaluation of support functions can be found, for instance, in [26, Section 13].

D. Set of Admissible Decision Variables

Within our setting, for a state $x \in \mathbb{R}^n$, the variables $\mathbf{z}_N := (z_0, \ldots, z_N)$, $\bar{\mathbf{s}}_N := (\bar{s}_0, \ldots, \bar{s}_N)$ and $\mathbf{v}_{N-1} := (v_0, \ldots, v_{N-1})$ determine entirely the state tubes $\mathbf{X}_N$ and the related control policy $\Pi_{N-1}(\cdot)$. However, in light of (2.8), each $\bar{s}_k$ can be expressed as follows

$$\forall k \in \mathbb{N}_N,\ \bar{s}_k = A_K^k(x - z_0). \tag{3.9}$$

Thus, for a state $x \in \mathbb{R}^n$, the actual underlying decision variable is

$$\mathbf{d}_N := (\mathbf{z}_N, \mathbf{v}_{N-1}) \in \mathbb{R}^{n_{\mathbf{d}_N}} \tag{3.10}$$

with $n_{\mathbf{d}_N} = N(n+m) + n$. The decision variable $\mathbf{d}_N$ is required to satisfy dynamical consistency constraints (2.6), tube initialization constraints (3.1), tube stage constraints (3.5) and tube terminal constraints (3.7), which are summarized as follows

$$\begin{aligned}
&\forall k \in \mathbb{N}_{N-1},\ z_{k+1} = Az_k + Bv_k, && (3.11a)\\
&\forall i \in \mathcal{I}_{\mathcal{X}_f},\ r_i^\top (x - z_0) \le 1, && (3.11b)\\
&\forall k \in \mathbb{N}_{N-1},\ \forall i \in \mathcal{I}_{\mathcal{Y}}, && \\
&\quad c_i^\top z_k + d_i^\top v_k + \eta_i^\top A_K^k (x - z_0) \le 1 - f_{(k,i)} \text{ and} && (3.11c)\\
&\forall i \in \mathcal{I}_{\mathcal{X}_f},\ r_i^\top z_N + r_i^\top A_K^N (x - z_0) \le 1 - g_i. && (3.11d)
\end{aligned}$$

For a state $x \in \mathbb{R}^n$, the set of admissible decision variables $\mathcal{D}_N(x)$ is given by

$$\mathcal{D}_N(x) := \{\mathbf{d}_N \ :\ \text{relations (3.11) hold}\}. \tag{3.12}$$

E. Cost

The utilized overall cost $V_N(\cdot) : \mathbb{R}^{n_{\mathbf{d}_N}} \to \mathbb{R}_{\ge 0}$ associated with the tube $\mathbf{X}_N$ is specified, for all $\mathbf{d}_N \in \mathbb{R}^{n_{\mathbf{d}_N}}$, by

$$V_N(\mathbf{d}_N) := \sum_{k=0}^{N-1} \left(z_k^\top Q z_k + v_k^\top R v_k \right) + z_N^\top P z_N. \tag{3.13}$$

Assumption 5: The matrices $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$ and $P \in \mathbb{R}^{n\times n}$ are all known, symmetric and positive definite, i.e., $Q = Q^\top \succ 0$, $R = R^\top \succ 0$ and $P = P^\top \succ 0$, and satisfy the following Lyapunov inequality

$$A_K^\top P A_K \preceq P - \left(Q + K^\top R K\right). \tag{3.14}$$

Within the considered setting, there exist scalars $\beta_1 \in (0, \infty)$ and $\beta_3 \in (0, \infty)$ such that, for all $\mathbf{d}_N \in \mathbb{R}^{n_{\mathbf{d}_N}}$, the overall cost $V_N(\mathbf{d}_N)$ satisfies

$$\begin{aligned}
&\beta_1 \|z_0\|^2 \le z_0^\top Q z_0,\ \beta_1 \|z_0\|^2 \le \beta_1 \|\mathbf{d}_N\|^2 \text{ and}\\
&\beta_1 \|\mathbf{d}_N\|^2 \le V_N(\mathbf{d}_N) \le \beta_3 \|\mathbf{d}_N\|^2.
\end{aligned} \tag{3.15}$$

IV. Improved Tube Model Predictive Control

A. Tube Optimal Control

For a current state $x \in \mathbb{R}^n$, the tube optimal control problem, labeled by $\mathfrak{P}_N(x)$, reduces to selection of $\mathbf{d}_N \in \mathcal{D}_N(x)$ that minimizes $V_N(\mathbf{d}_N)$ so that, for all $x \in \mathbb{R}^n$,

$$V_N^0(x) := \min_{\mathbf{d}_N} \{V_N(\mathbf{d}_N) \ :\ \mathbf{d}_N \in \mathcal{D}_N(x)\} \text{ and} \tag{4.1}$$

$$\mathbf{d}_N^0(x) := \arg\min_{\mathbf{d}_N} \{V_N(\mathbf{d}_N) \ :\ \mathbf{d}_N \in \mathcal{D}_N(x)\}. \tag{4.2}$$

Here, $\mathbf{d}_N$ is defined in (3.10), $V_N(\mathbf{d}_N)$ is defined in (3.13), and $\mathcal{D}_N(x)$ is specified by (3.12). The effective domain $\mathcal{C}_N$ of the value function $V_N^0(\cdot)$ and its optimizer $\mathbf{d}_N^0(\cdot)$, also known as the N–step controllable set, is

$$\mathcal{C}_N := \{x \ :\ \mathcal{D}_N(x) \ne \emptyset\}. \tag{4.3}$$

In this setting, the cost function $\mathbf{d}_N \mapsto V_N(\mathbf{d}_N)$ is a strictly convex and quadratic function. The set of admissible decision variables $\mathcal{D}_N(x)$, for each $x \in \mathcal{C}_N$, is a nonempty, closed polyhedral subset of $\mathbb{R}^{n_{\mathbf{d}_N}}$. Thus, for any given $x \in \mathbb{R}^n$, $\mathfrak{P}_N(x)$ is a strictly convex quadratic programming problem, which is feasible for all $x \in \mathcal{C}_N$. The key properties of the value function $V_N^0(\cdot)$, its optimizer $\mathbf{d}_N^0(\cdot)$, and their effective domain $\mathcal{C}_N$ can be summarized by the following.

Theorem 1: Suppose Assumptions 1–5 hold, and take any $N \in \mathbb{N}$.

- The value function $V_N^0(\cdot)$ is continuous, convex, piecewise quadratic and such that:

$$\forall x \in \mathcal{X}_f,\ V_N^0(x) = 0, \text{ and} \tag{4.4}$$

$$\forall x \in \mathcal{C}_N \setminus \mathcal{X}_f,\ 0 < V_N^0(x) < \infty. \tag{4.5}$$

- The optimizer $\mathbf{d}_N^0(\cdot)$ is continuous, piecewise affine and such that:

$$\forall x \in \mathcal{X}_f,\ \left\|\mathbf{d}_N^0(x)\right\| = 0 \text{ and} \tag{4.6}$$

$$\forall x \in \mathcal{C}_N \setminus \mathcal{X}_f,\ \left\|\mathbf{d}_N^0(x)\right\| > 0 \text{ and } \|z_0^0(x)\| > 0. \tag{4.7}$$

- The effective domain $\mathcal{C}_N \subseteq \mathbb{R}^n$ is a polyhedral proper D–set and satisfies

$$\mathcal{X}_f = \mathcal{C}_0 \subseteq \ldots \subseteq \mathcal{C}_N \subseteq \mathcal{Y}_x, \tag{4.8}$$

where $\mathcal{Y}_x$ is the $(x, u) \mapsto x$ projection of the set $\mathcal{Y}$.

B. Tube MPC

The tube model predictive control evaluates implicitly the control law $\kappa_N(\cdot)$, specified by

$$\forall x \in \mathcal{C}_N,\ \kappa_N(x) := v_0^0(x) + K\left(x - z_0^0(x)\right), \tag{4.9}$$

and it induces the tube model predictive controlled uncertain dynamics

$$\forall x \in \mathcal{C}_N,\ x^+ \in Ax + B\kappa_N(x) \oplus \mathcal{W}. \tag{4.10}$$

In view of (4.6) in Theorem 1, for any $x \in \mathcal{X}_f$, $z_0^0(x) = 0$ and $v_0^0(x) = 0$. Thus, for all states x in the terminal constraint set $\mathcal{X}_f$, the tube model predictive control law $\kappa_N(\cdot)$ and its controlled uncertain dynamics (4.10) satisfy

$$\forall x \in \mathcal{X}_f,\ \kappa_N(x) = Kx \text{ and } x^+ \in A_K x \oplus \mathcal{W}. \tag{4.11}$$

C. Robust Positive Invariance and Stability

In our setting, Theorem 2 establishes the robust positive invariance property of the effective domain $\mathcal{C}_N$ (also known as the robust recursive feasibility of $\mathfrak{P}_N(\cdot)$).

Theorem 2: Suppose Assumptions 1–5 hold, and take any $N \in \mathbb{N}$. The effective domain $\mathcal{C}_N$ is a robust positively invariant set for the dynamics (4.10) and the implicitly induced constraints $(x, \kappa_N(x)) \in \mathcal{Y}$.

In our setting, the lower and upper bounds of the overall cost $V_N(\cdot)$ in (3.15) and Theorem 1 yield the following.

Proposition 3: Suppose Assumptions 1–5 hold, and take any $N \in \mathbb{N}$. There exist two scalars $\beta_1 \in (0, \infty)$ and $\beta_2 \in (0, \infty)$ such that, for all $x \in \mathcal{C}_N$,

$$\beta_1 \|z_0^0(x)\|^2 \leq V_N^0(x) \leq \beta_2 \|z_0^0(x)\|^2, \tag{4.12}$$

and, for all $x \in \mathcal{C}_N$ and all $x^+ \in Ax + B\kappa_N(x) \oplus \mathcal{W}$,

$$V_N^0(x^+) \leq V_N^0(x) - \beta_1 \|z_0^0(x)\|^2. \tag{4.13}$$

The main, and relatively direct, ramification of the Proposition 3 and Theorems 1 and 2 is the following.

Theorem 3: Suppose Assumptions 1–5 hold, and take any $N \in \mathbb{N}$. The maximal robust positively invariant set $\mathcal{X}_f$ is robustly exponentially stable for the dynamics (4.10) with the domain of attraction being equal to $\mathcal{C}_N$.

When A_K satisfies Assumption 2, it is well-known that the minimal robust positively invariant set $\mathcal{X}_\mathcal{O}$ is robustly exponentially stable for the uncertain dynamics $x^+ \in A_K x \oplus \mathcal{W}$ with the domain of attraction being equal to the maximal robust positively invariant set $\mathcal{X}_f$. In light of (4.11), Theorem 3 and a local application of the analysis in [27], the following refinement of the robust exponential stability is affirmative.

Corollary 1: Suppose Assumptions 1–5 hold, and take any $N \in \mathbb{N}$. The set $\mathcal{X}_\mathcal{O}$ is robustly exponentially stable for (4.10) with the domain of attraction being equal to $\mathcal{C}_N$.

V. Discussion

A. Comparison

State Decomposition and Control Policy: Unlike the proposal in this paper, which decomposes the state predictions into three components, [11]–[13] decompose the state predictions into two components $x_k = z_k + s_k$ for all $k \in \mathbb{N}_{N-1}$, with s_k denoting the local uncertainty. The proposed method and [12], [13] use the control parameterization (2.4), whereas the control parameterization of [11] is defined as $u_k = Kx_k + \mu_k$ for all $k \in \mathbb{N}_{N-1}$, where $\mu_k \in \mathbb{R}^m$ is a control offset. However, the latter can be recovered by a direct algebraic change of variables as $\mu_k = v_k - Kz_k$ for all $k \in \mathbb{N}_{N-1}$.

Cost Function: The proposal in this paper and [12], [13] penalize the nominal state and control predictions, as defined in (3.13). Proposal [11] penalizes the control offsets, given by

$$J_N(\boldsymbol{\mu}_{N-1}) = \sum_{k=0}^{N-1} \mu_k^\top \Psi \mu_k = \sum_{k=0}^{N-1} (v_k - Kz_k)^\top \Psi (v_k - Kz_k),$$

where $\boldsymbol{\mu}_{N-1} := (\mu_0, \ldots, \mu_{N-1})$ and $\Psi = \Psi^\top \succ 0$. Within our setting, if K and P are such that (3.14) holds true with equality. Then, for $\Psi = R + B^\top PB$, it holds that $J_N(\boldsymbol{\mu}_{N-1}) = V_N(\mathbf{d}_N) - z_0^\top Q z_0$. However, $J_N(\cdot)$ discards the cost term $z_0^\top Q z_0$ with $z_0 = x$ from $V_N(\cdot)$, which results in a value function $J_N^0(\cdot)$ that is not guaranteed to admit adequate upper bound necessary for establishing stronger robust exponential stability properties. Consequently, [11] only establishes robust convergence results with respect to $\mathcal{X}_\mathcal{O}$ albeit the authors in [28] proved that the tube MPC proposal of [11] is robust asymptotically stable.

Initialization: Proposal [11] enforces an initial condition $z_0 = x$. This is a disadvantage of [11] and, in fact, the main cause of the lack of robust stability guarantees. Proposal [12] enforces an initial condition $z_0 = z_1^\star$, where $z_1^\star$ is the predicted nominal state one-step ahead of the previous time instant. Proposal [13] allows for an initial condition $x - z_0 \in \mathcal{S}$, where $\mathcal{S}$ is an outer invariant approximation of the minimal robust positively invariant set for $s^+ = A_K s + w$, $w \in \mathcal{W}$. This initial condition facilitates a direct geometric argument for robust exponential stability of the set $\mathcal{S}$. In this regard, the proposal in this paper refines all proposals [11]–[13] by allowing for a relaxed initial condition $x - z_0 \in \mathcal{X}_f$.

Tubes Stage Constraints: Proposals [12], [13] make use of tubes with rigid cross-section $\mathcal{S}$, while the proposal in this paper and [11] make use of tubes with time-varying cross-sections $\widetilde{S}_k$ for all $k \in \mathbb{N}_N$. Tubes with time-varying cross-sections are less conservative since $\widetilde{S}_k \subseteq \mathcal{S}$ for all $k \in \mathbb{N}_N$. Thus, the utilization of time-varying cross-sections $\widetilde{S}_k$ reduces the effects of the uncertainty on the constraints. In order to modify stage constraints, for all $i \in \mathcal{I}_\mathcal{Y}$, the support function $\mathrm{h}\left(\widetilde{S}_k, \eta_i\right)$ needs to be evaluated offline for all $k \in \mathbb{N}_{N-1}$, as shown in (3.6), while for all $i \in \mathcal{I}_\mathcal{Y}$, each support function $\mathrm{h}(\mathcal{S}, \eta_i)$ needs to be evaluated offline only once. In this regard, a concrete comparison of the involved computation cost depends on the horizon N and the underlying representations of sets $\widetilde{S}_k$ and $\mathcal{S}$.

Tubes Terminal Constraints: The terminal constraints in this paper and [11] are specified by $z_N + \bar{s}_N \oplus \widetilde{S}_N \subseteq \mathcal{X}_f$ and $z_N \oplus \widetilde{S}_N \subseteq \mathcal{X}_f$, respectively. Both proposals [12] and [13] make use of terminal constraints $z_N \in \mathcal{Z}_f$, where $\mathcal{Z}_f$ is the maximal positively invariant set for system $z^+ = (A+BK_f)z$ and constraints $z \in \{z : \forall s \in \mathcal{S},\ (z+s, K_f(z+s)) \in \mathcal{Y}\}$. Note that in [12], [13], K_f is set to be the same as K, but it is possible to choose a matrix K that is different from K_f, as pointed out in [29]. When $K_f = K$, it is not difficult to verify that $\mathcal{Z}_f \oplus \mathcal{S} \subseteq \mathcal{X}_f$ since both sets $\mathcal{Z}_f \oplus \mathcal{S}$ and $\mathcal{X}_f$ are robust positively invariant [29] and $\mathcal{X}_f$ is the maximal one. Since for all $N \in \mathbb{N}$, $\widetilde{S}_N \subseteq \mathcal{S}$, it holds that $\mathcal{Z}_f \oplus \widetilde{S}_N \subseteq \mathcal{Z}_f \oplus \mathcal{S} \subseteq \mathcal{X}_f$. Thus, compared to [12], [13], terminal constraints used in this paper and [11] bring about a further reduction of the effects of the uncertainty on the constraints.

Remark 2: Tuning the control feedback K and the local terminal feedback K_f separately for [12], [13] provides more flexibility and may result in a larger effective domain. In [20], the authors have presented a scheme similar to [11] where K is not necessarily equal to K_f. However, while K_f obtained from the solution to the unconstrained infinite horizon linear quadratic regulator (A, B, Q, R) is optimal, a systematic way for optimally selecting matrix K to simultaneously enlarge the

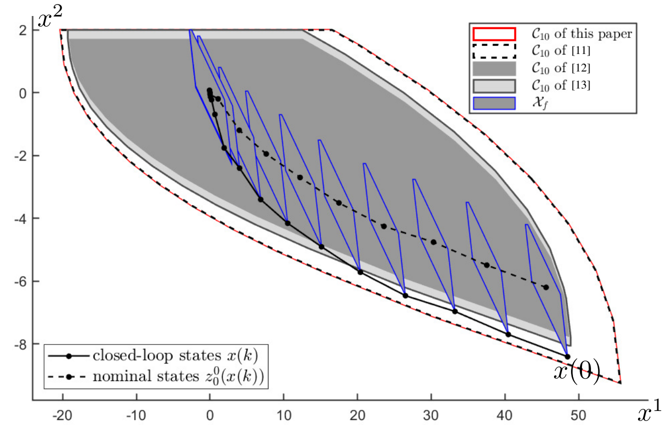

Fig. 1. Effective domains and closed-loop simulations.

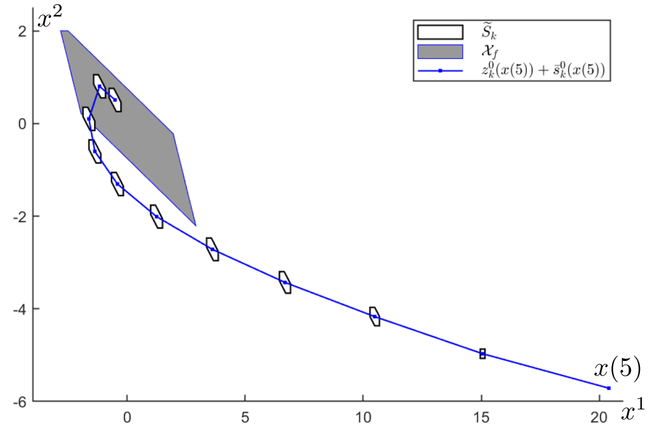

Fig. 2. The predicted tube $\mathbf{X}_N$ at sampling instant 5.

effective domain and reduce the value function requires further analysis, and it deserves study in its own right.

Online Computation: The tube optimal control problems of this paper and [11]–[13] are strictly convex quadratic programming. The number of decision variables and constraints of these four optimization problems are listed in Table 1, in which the state and control constraints were uniformly considered as in (2.3) and the nominal dynamics were not eliminated to maintain the sparse structure of the tube optimal control problems. The notations $n_{\mathcal{Z}_f}$ and $n_{\mathcal{S}}$ denote the number of the inequality representations of the sets $\mathcal{Z}_f$ and $\mathcal{S}$, respectively. As shown in Table 1, the numbers of decision variables and constraints scale linearly with respect to horizon length N for all proposals, indicating that the online computations of all proposals are of the same order of complexity. As also shown in Table 1, the online computations are almost identical for all proposals, especially when N is large.

TABLE 1. Numbers of Decision Variables and Constraints

method	decision variables	affine equalities	affine inequalities
This paper	$N(n+m)+n$	Nn	$Nn_{\mathcal{Y}}+2n_f$
[11]	Nm	$N(n+m)$	$Nn_{\mathcal{Y}}+n_f$
[12]	$N(n+m)$	Nn	$Nn_{\mathcal{Y}}+n_{\mathcal{Z}_f}$
[13]	$N(n+m)+n$	Nn	$Nn_{\mathcal{Y}}+n_{\mathcal{S}}+n_{\mathcal{Z}_f}$

B. Numerical Illustrations

Example 1: Our first example is taken from the constrained double integrator considered in [13], with the only exception that the stage constraint set is changed to $\mathcal{Y} := \mathbb{X} \times \mathbb{U}$ and

$$\mathbb{X} := \{x \in \mathbb{R}^2 \ : \ \|x\|_\infty \le 100 \text{ and } [0,1]x \le 2\} \text{ and}$$
$$\mathbb{U} := \{u \in \mathbb{R} \ : \ -1 \le u \le 1\}.$$

The terminal weighting matrix P and the feedback gain matrix K are obtained as the solutions to the infinite horizon unconstrained optimal control for (A, B, Q, R).

We choose a prediction horizon $N = 10$ and start from an initial state $x(0) = [48.5; -8.4]$. In Fig. 1, the effective domains of the proposed method and [11] are identical, and they include the effective domains of [12], [13]. Starting from $x(0)$ with a sampled disturbance sequence, the closed-loop state $x(k)$ converges to the maximal robust positively invariant set $\mathcal{X}_f$ in 11 steps and to the minimal robust positively invariant set $\mathcal{X}_{\mathcal{O}}$ in 12 steps. Fig. 2 visualizes the predicted tube $\mathbf{X}_{10}$ at sampling instant 5, and it shows that the required tube constraints are satisfied. We start from an initial state $x(0) = [48; -7]$ and simulate the control processes with 1000 times for [12], [13] and the proposal in this paper. Fig. 3 shows the averaged value functions, labeled by $\bar{V}_{10}^0(\cdot)$, illustrating that the proposed method achieves the lowest cost and faster convergence compared to [12], [13]. We remark that the cost function $J_N(\cdot)$ of [11] is defined on the control offsets $\boldsymbol{\mu}_{N-1}$, as discussed in Section V-A. Thus, we do not include the comparison of the corresponding results of [11] in Fig. 3 for the sake of fairness.

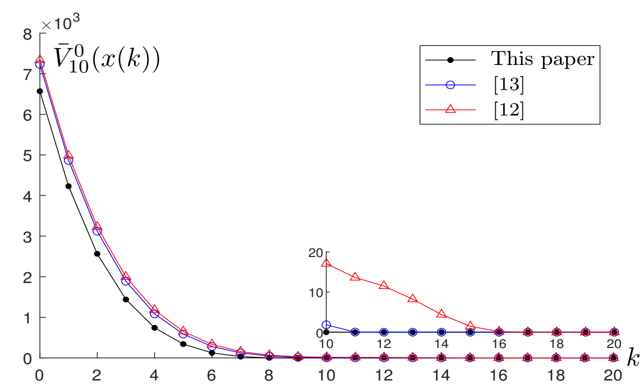

Fig. 3. Averaged value functions $\bar{V}_{10}^0(\cdot)$ along $\{x(k)\}_{k\in\mathbb{N}_{20}}$.

To highlight the benefits of using support functions, we consider the following example in higher dimensions.

Example 2: Our second example is taken from [14, Section 6.2], which is a variation of a modern transport airplane model in [30, Example AC9]. In this example, $n = 10$ and $m = 4$. The discrete time system matrices A and B are obtained via the Euler discretization with sampling period $T = 0.5$ [s]. The relevant sets $\mathcal{Y}$ and $\mathcal{W}$, and matrices Q and R are given by, $\mathcal{Y} = 500\mathcal{B}_\infty^{10} \times 50\mathcal{B}_\infty^4$, $\mathcal{W} = \mathcal{B}_2^{10}$, $Q = 100I$ and $R = I$. The terminal weighting matrix P and the feedback gain matrix K are also obtained as the solutions to the infinite horizon

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unconstrained optimal control for (A, B, Q, R). The prediction horizon is specified by $N = 20$.

For offline implementation in MATLAB, the maximal robust positively invariant set $\mathcal{X}_f$ is computed in 0.55 [s], and all the relevant scalars $f_{(k,i)}$ and g_i are computed in 0.008 [s] by using NORM function. Thus, the offline design is performed successfully in less than 0.56 [s]. For online implementation in MATLAB with QUADPROG solver, we start the simulation from an initial state $x(0) = [8; 50; 50; 100; 7; 15; 100; 2; 100; 50]$ for 20 time steps with a randomly sampled disturbance sequence. The average time to solve each of the considered quadratic programming problems is 0.08 [s]. Fig. 4 depicts the nominal state sequence $\{z_0^0(x(k))\}_{k\in\mathbb{N}_{19}}$ and the corresponding closed-loop state trajectory $\{x(k)\}_{k\in\mathbb{N}_{20}}$, illustrating that the proposed tube model predictive controlled states converge exponentially fast to the associated minimal robust positively invariant set $\mathcal{X}_{\mathcal{O}}$.

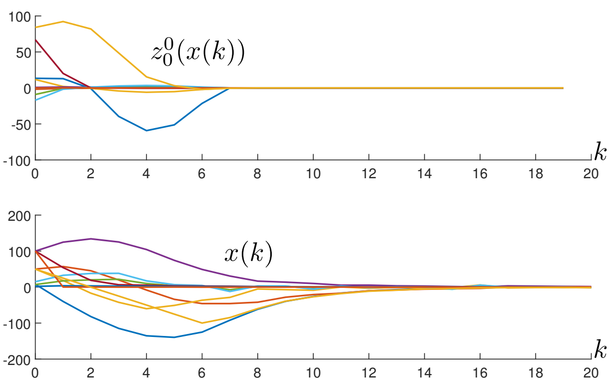

Fig. 4. Sequences: $\{z_0^0(x(k))\}_{k\in\mathbb{N}_{19}}$ and $\{x(k)\}_{k\in\mathbb{N}_{20}}$.

C. Concluding Remarks

This article has revisited the early tube MPC methods [11]–[13] and presented a refined tube MPC for more general stage, mixed state and control constraint set $\mathcal{Y}$ and disturbance set $\mathcal{W}$. The refined tube MPC preserves all desirable computational and structural properties of its predecessors, and it also improves the desirable control-theoretic properties of [11]–[13] to a reasonable extent and simplifies the offline computations with the help of support functions. An important direction for future research involves extending the proposed refinement to incorporate a wider spectrum of control systems and adopting less conservative tube parameterizations.

APPENDIX

Proof of Propositions 1 and 2. These claims can be verified by directly applying [14, Proposition 1]. Specifically, Proposition 1 of [14] applies by replacing the underlying support functions therein with the support functions $\mathrm{h}\left(\widetilde{S}_k, \cdot\right)$ for all $k \in \mathbb{N}_N$. In regard of $\widetilde{S}_k$ specified in (2.11), the equations (3.6) and (3.8) follow from [14, Lemma 2].

Proof of Theorem 1. The related topological statements follow from the standard properties of the solution to the considered parametric convex quadratic programming problem $\mathfrak{P}_N(x)$ and its construction, see [2], [13], [31] for more details. Since $x \in \mathcal{X}_f$, $z_N(x) = \ldots = z_0(x) = 0$ and $v_{N-1}(x) = \ldots = v_0(x) = 0$ are feasible for $\mathfrak{P}_N(x)$. Hence $0 \le V_N^0(x) \le V_N(\mathbf{d}_N(x)) = 0$ with $\mathbf{d}_N(x) = \mathbf{0}$, which ensures that (4.4) and (4.6) hold. Likewise, if $x \in \mathcal{C}_N \setminus \mathcal{X}_f$, the constraint $x - z_0(x) \in \mathcal{X}_f$ implies that $z_0(x) \neq 0$ so that (4.5) and (4.7) hold. The set-inclusion relation (4.8) holds true by applying [2, Proposition 2.10].

Proof of Theorem 2. When $N = 0$, the statement holds directly by the robust positive invariance of the set $\mathcal{X}_f$ for the controlled dynamics (4.11). For any $N \in \mathbb{N}_+$, suppose $x \in \mathcal{C}_N$ so that $\mathfrak{P}_N(x)$ is feasible and the optimizer $\mathbf{d}_N^0(x) = \left(\mathbf{z}_N^0(x),\, \mathbf{v}_{N-1}^0(x)\right)$ exists. For all $w \in \mathcal{W}$, it holds that $x^+ = Ax + B\kappa_N(x) + w = z_1^0(x) + A_K(x - z_0^0(x)) + w$ so that $x^+ - z_1^0(x) = A_K(x - z_0^0(x)) + w$. Now, we consider

$$\begin{aligned} \mathbf{d}_N(x^+) &:= \left(\mathbf{z}_N(x^+),\, \mathbf{v}_{N-1}(x^+)\right), \text{ with} \\ \mathbf{z}_N(x^+) &= \left(z_1^0(x), \ldots, z_N^0(x), A_K z_N^0(x)\right) \text{ and} \\ \mathbf{v}_{N-1}(x^+) &= \left(v_1^0(x), \ldots, v_{N-1}^0(x), K z_N^0(x)\right), \end{aligned} \tag{A.1}$$

and establish that $\mathbf{d}_N(x^+)$ is feasible for $\mathfrak{P}_N(x^+)$, i.e., $\mathbf{d}_N(x^+) \in \mathcal{D}_N(x^+)$. The dynamic consistency constraints (3.11a) are satisfied by the construction of $\mathbf{d}_N(x^+)$. The tube initialization constraint $x^+ - z_0(x^+) = x^+ - z_1^0(x) = A_K(x - z_0^0(x)) + w \in \mathcal{X}_f$ holds for all $w \in \mathcal{W}$. This is due to the fact that $(x - z_0^0(x)) \in \mathcal{X}_f$ and $\mathcal{X}_f$ is robust positively invariant as specified in Assumption 3. Thus, the inequality constraints (3.11b) is satisfied.

For all $k \in \mathbb{N}_{N-1}$, all $w \in \mathcal{W}$ and all $\tilde{s} \in \widetilde{S}_k$, $z_k(x^+) + A_K^k(x^+ - z_1^0(x)) + \tilde{s} = z_{k+1}^0(x) + A_K^{k+1}(x - z_0^0(x)) + A_K^k w + \tilde{s}$ and $v_k(x^+) + KA_K^k(x^+ - z_1^0(x)) + K\tilde{s} = v_{k+1}^0(x) + KA_K^{k+1}(x - z_0^0(x)) + KA_K^k w + K\tilde{s}$. It follows from (2.11) that for all $w \in \mathcal{W}$ and all $\tilde{s} \in \widetilde{S}_k$, $A_K^k w + \tilde{s} \in \widetilde{S}_{k+1}$. Therefore, for all $k \in \mathbb{N}_{N-2}$ and all $\hat{s} \in \widetilde{S}_{k+1}$, $(z_{k+1}^0(x) + A_K^{k+1}(x - z_0^0(x)) + \hat{s}, v_{k+1}^0(x) + KA_K^{k+1}(x - z_0^0(x)) + K\hat{s}) \in \mathcal{Y}$ by the definition of stage constraints (3.5). For $k = N-1$ and all $\hat{s} \in \widetilde{S}_N$, $z_{k+1}^0(x) + A_K^{k+1}(x - z_0^0(x)) + \hat{s} \in \mathcal{X}_f$ holds by the definition of terminal constraints (3.7). Then, by robust positive invariance property of $\mathcal{X}_f$, $(z_N^0(x) + A_K^N(x - z_0^0(x)) + \hat{s}, Kz_N^0(x) + KA_K^N(x - z_0^0(x)) + K\hat{s}) \in \mathcal{Y}$ holds. Thus, for all $k \in \mathbb{N}_{N-1}$ and all $\tilde{s} \in \widetilde{S}_k$, we have that $(z_k(x^+) + A_K^k(x^+ - z_1^0(x)) + \tilde{s}, v_k(x^+) + KA_K^k(x^+ - z_1^0(x)) + K\tilde{s}) \in \mathcal{Y}$, i.e., the inequality constraints (3.11c) are satisfied.

For $k = N$, $z_N(x^+) + A_K^N(x^+ - z_1^0(x)) \oplus \widetilde{S}_N = A_K z_N^0(x) + A_K^{N+1}(x - z_0^0(x)) + A_K^N w \oplus \widetilde{S}_N \subseteq A_K z_N^0(x) + A_K^{N+1}(x - z_0^0(x)) \oplus \widetilde{S}_{N+1}$. It follows from the definition of constraints (3.7) that $z_N^0(x) + A_K^N(x - z_0^0(x)) \oplus \widetilde{S}_N \subseteq \mathcal{X}_f$. Then, by the robust invariance property of $\mathcal{X}_f$, it holds that $A_K(z_N^0(x) + A_K^N(x - z_0^0(x)) \oplus \widetilde{S}_N) \oplus \mathcal{W} = A_K z_N^0(x) + A_K^{N+1}(x - z_0^0(x)) \oplus \widetilde{S}_{N+1} \subseteq \mathcal{X}_f$. Thus, $z_N(x^+) + A_K^N(x^+ - z_1^0(x)) \oplus \widetilde{S}_N \subseteq \mathcal{X}_f$ holds, i.e., the inequality constraints (3.11d) are satisfied. Hence, we have that $\mathbf{d}_N(x^+) \in \mathcal{D}_N(x^+)$, which further implies that for all $w \in \mathcal{W}$, $x^+ = Ax + B\kappa_N(x) + w \in \mathcal{C}_N$ and completes the proof.

Proof of Proposition 3 and Theorem 3. The admissible decision variable $\mathbf{d}_N(x^+)$ specified in (A.1) yields the desired cost decrease, i.e.,

$$\begin{aligned}\forall x \in \mathcal{C}_N,\ V_N^0(x^+) - V_N^0(x) &\leq V_N(\mathbf{d}_N(x^+)) - V_N^0(x)\\ &= \|z_N^0(x)\|_Q^2 + \|v_N^0(x)\|_R^2 + \|A_K z_N^0(x)\|_P^2\\ &\quad - \|z_N^0(x)\|_P^2 - \|z_0^0(x)\|_Q^2 - \|v_0^0(x)\|_R^2\\ &\leq -\|z_0^0(x)\|_Q^2 - \|v_0^0(x)\|_R^2,\end{aligned}$$

where the last inequality holds due to Assumption 5. Since there exists $\beta_1 > 0$ such that $\beta_1\|z_0^0(x)\|^2 \leq \|z_0^0(x)\|_Q^2 + \|v_0^0(x)\|_R^2$, it follows that

$$\forall x \in \mathcal{C}_N,\ V_N^0(x^+) - V_N^0(x) \leq -\beta_1\|z_0^0(x)\|^2. \tag{A.2}$$

It follows from (3.15) that, for all $x \in \mathcal{C}_N$, there also exists $\beta_3 > 0$ such that

$$\beta_1\|z_0^0(x)\|^2 \leq V_N^0(x) \leq \beta_3\|\mathbf{d}_N^0(x)\|^2. \tag{A.3}$$

The proof can now be completed by adapting an argument used in the proof of [32, Theorem 3]. Namely, in light of Theorem 1, the optimizer $\mathbf{d}_N^0(\cdot)$ is Lipschitz continuous. Thus, we have that for all $x \in \mathcal{C}_N$ and all $y \in \mathcal{C}_N$,

$$\|\mathbf{d}_N^0(x) - \mathbf{d}_N^0(y)\| \leq L\|x - y\|, \tag{A.4}$$

in which $L > 0$ denotes the associated Lipschitz constant. For any $x \in \mathcal{C}_N$, by construction of $y := x - z_0^0(x) \in \mathcal{X}_f \subseteq \mathcal{C}_N$, we have $\mathbf{d}_N^0(y) = \mathbf{0}$ in light of (4.6). Then, (A.4) yields that

$$\forall x \in \mathcal{C}_N,\ \|\mathbf{d}_N^0(x)\| \leq L\|z_0^0(x)\|. \tag{A.5}$$

It follows from (A.3) and (A.5) that, with $\beta_2 := \beta_3 L^2$,

$$\forall x \in \mathcal{C}_N,\ \beta_1\|z_0^0(x)\|^2 \leq V_N^0(x) \leq \beta_2\|z_0^0(x)\|^2. \tag{A.6}$$

Thus, in view of (A.6) and (A.2), Proposition 3 is proved. Additionally, in light of [2, Theorem B.19], Theorems 1–2 and Proposition 3 yield the statements of Theorem 3.

Proof of Corollary 1. Let the set sequence $\{\mathcal{Q}_k \subseteq \mathbb{R}^n\}_{k\in\mathbb{N}}$ be generated by the following set-dynamics

$$\forall k \in \mathbb{N},\ \mathcal{Q}_{k+1} = A_K\mathcal{Q}_k \oplus \mathcal{W} \text{ with } \mathcal{Q}_0 \subseteq \mathcal{X}_f.$$

Proposition 4.3 of [27] directly implies that, for any set $\mathcal{Q}_0$ in the set of compact subsets of $\mathcal{X}_f$, $\{\mathcal{Q}_k\}_{k\in\mathbb{N}}$ converges exponentially fast to $\mathcal{X}_\mathcal{O}$ in a stable manner with respect to the Hausdorff distance. This fact in conjunction with Theorem 3 implies that, the set $\mathcal{X}_\mathcal{O}$ is robustly exponentially stable for (4.10) with domain of attraction being equal to $\mathcal{C}_N$.

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