Recursive Feasibility of Stochastic Model Predictive Control with Mission-Wide Probabilistic Constraints

Kai Wang, Sébastien Gros

Department of Engineering Cybernetics
Norwegian University of Science and Technology (NTNU)

NTNU

Overview

Background

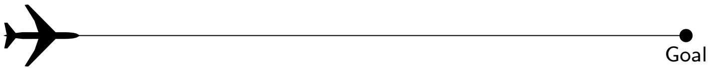
- Stochastic Optimal Control
- Mission-Wide Chance-Constrained Optimal Control
- Chance Constraints: mission-wide & stage-wise

Contributions

- Remaining MWPSs & Initially Prescribed MWPS
  (MWPS: Mission-Wide Probability of Safety)
- Recursive Feasibility of MWPS Constraints
- Stochastic Linear MPC with MWPS guarantee

Stochastic Optimal Control

Mission: plan an optimal trajectory for the drone



State transition: $$\underbrace{\mathbf{s}, \mathbf{u}}_{\text{current state\&input}} \xrightarrow{\text{s.t. disturbance } \mathbf{w}} \underbrace{\mathbf{s}_+}_{\text{successor state}}$$

- $\mathbf{s}, \mathbf{u}$ and $\mathbf{w}$ are all continuous
- we will describe this transition via conditional distribution:

$$\rho\left[\mathbf{s}_+ \,|\, \mathbf{s}, \mathbf{u}\right] \qquad \text{(depending on the distribution of } \mathbf{w}\text{)}$$

- In classic control, it is modeled as $\mathbf{s}_+ = \mathbf{f}(\mathbf{s}, \mathbf{u}, \mathbf{w})$ for some function $\mathbf{f}$

Stochastic Optimal Control

Mission: plan an optimal trajectory for the drone within N stages

A simplified diagram above: the drone may reach the goal point before stage N.

Performance assessment:

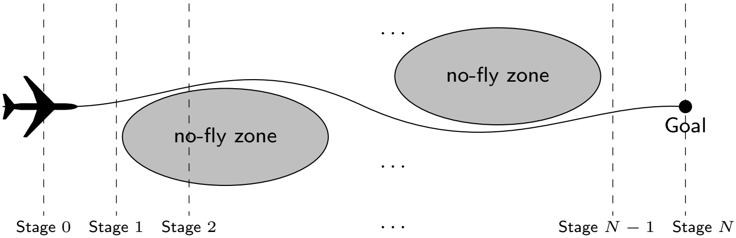
- N: mission horizon or number of times control is applied (typically very large)
- stage cost: $L(\mathbf{s}, \mathbf{u}) \in \mathbb{R}$, terminal cost: $M(\mathbf{s}_N) \in \mathbb{R}$
- policy sequence is $\boldsymbol{\pi} := \{\boldsymbol{\pi}_0, \ldots, \boldsymbol{\pi}_{N-1}\}$ such that $\mathbf{u}_k = \boldsymbol{\pi}_k(\mathbf{s}_k)$
- total cost:

$$\mathbb{E}\left[M(\mathbf{s}_N) + \sum_{k=0}^{N-1} L(\mathbf{s}_k, \boldsymbol{\pi}_k(\mathbf{s}_k))\right]$$

(the expectation is taken over the state trajectories)

Mission-Wide Chance-Constrained Optimal Control

Mission: plan an optimal trajectory for the drone within N stages and subject to safety constraints **Mission:** plan an optimal trajectory for the drone within N stages and subject to probabilistic safety constraints



Safety considerations:

- A mission to be safe if: $\mathbf{s}_{1,\dots,N} \in \mathbb{S}$.(may be impossible!!!)
- Mission-Wide Probability of Safety (MWPS):

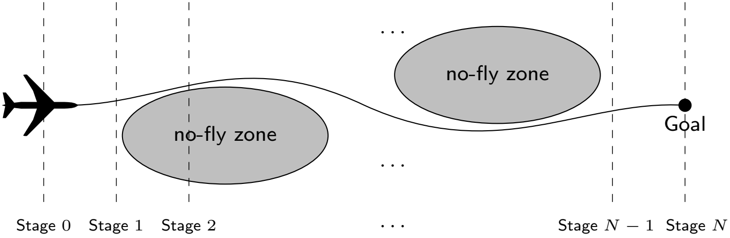
$$\mathbb{P}[\mathbf{s}_{1,\dots,N} \in \mathbb{S} \,|\, \mathbf{s}_0, \boldsymbol{\pi}]$$

- MWPS constraint, i.e., mission-wide chance constraint:

$$\mathbb{P}[\mathbf{s}_{1,\dots,N} \in \mathbb{S} \,|\, \mathbf{s}_0, \boldsymbol{\pi}] \geq S, \quad \text{with } S \in [0, 1]$$

Mission-Wide Chance-Constrained Optimal Control

Mission: plan an optimal trajectory for the drone within N stages and subject to MWPS constraint

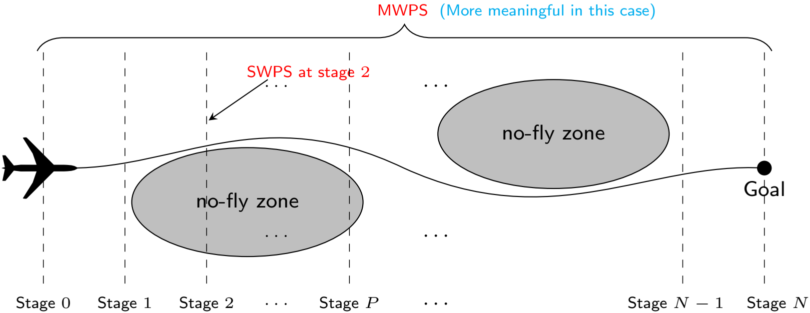


Find a policy sequence solution of:

$$\begin{aligned} \min_{\boldsymbol{\pi}} \quad & \mathbb{E}\left[M(\mathbf{s}_N) + \sum_{k=0}^{N-1} L(\mathbf{s}_k, \boldsymbol{\pi}_k(\mathbf{s}_k))\right] \\ \text{s.t.} \quad & \mathbb{P}[\mathbf{s}_{1,\ldots,N} \in \mathbb{S} \,|\, \mathbf{s}_0, \boldsymbol{\pi}] \geq S \end{aligned}$$

Notice: the space of policies is functional, and dynamic programming fails to apply directly.

Chance Constraints: mission-wide & stage-wise



(In engineering, we typically care about the probability that the mission is successful, i.e., MWPS.)

Comparison:

- Mission-Wide Probability of Safety (MWPS): w.r.t. the whole mission.
- Stage-Wise Probability of Safety (SWPS): w.r.t. a single time stage.
- Classic SMPC that enforces P joint SWPSs, P is the prediction horizon (typically $P \ll N$)

It is hard to non-conservatively enforce MWPS via (multiple) SWPS because of the correlation between successive states

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  (MWPS: Mission-Wide Probability of Safety)
- Recursive Feasibility of MWPS Constraints
- Stochastic Linear MPC with MWPS guarantee
  (A tentative solution given by shrinking-horizon policies)

Remaining MWPSs & Initially Prescribed MWPS

Once a policy sequence $\boldsymbol{\pi} := \{\boldsymbol{\pi}_0, \ldots, \boldsymbol{\pi}_{N-1}\}$ is selected in the beginning of the mission.

- initially prescribed MWPS (fixed value):

  $$\mathbb{P}_{0,N} := \mathbb{P}\left[\mathbf{s}_{1,\ldots,N} \in \mathbb{S} \,\middle|\, \mathbf{s}_0, \boldsymbol{\pi}\right]$$

- remaining MWPSs (random value):
  (depend on the specific realization of $\mathbf{s}_{1,\ldots,k}$)

  $$\mathbb{P}_{k,N} := \mathbb{P}\left[\mathbf{s}_{k+1,\ldots,N} \in \mathbb{S} \,\middle|\, \mathbf{s}_k, \{\boldsymbol{\pi}_k, \ldots, \boldsymbol{\pi}_{N-1}\}\right]$$

Any relations between $\mathbb{P}_{0,N}$ and $\mathbb{P}_{k,N}$?

Lemma 1: Given a policy sequence $\boldsymbol{\pi}$, we observe that

$$\mathbb{E}_{\{\mathbf{s}_{1,\ldots,k} \in \mathbb{S} \,|\, \mathbf{s}_0, \{\boldsymbol{\pi}_0, \ldots, \boldsymbol{\pi}_{k-1}\}\}}\left[\, \mathbb{P}_{k,N} \,\right] = \mathbb{P}_{0,N}$$

for all $k = 1, \ldots, N-1$, i.e. the remaining MWPSs are equal to the initially prescribed MWPS in the expected value sense.

This Lemma forms a basis for constructing a recursively feasible SMPC controller

Recursive Feasibility of MWPS Constraints

SMPC with Shrinking-horizon policy (solving N optimal control problems in an "on-line" fashion)

- Find policy

$$\tilde{\boldsymbol{\pi}}^k = \left\{ \tilde{\boldsymbol{\pi}}_k^k, \ldots, \tilde{\boldsymbol{\pi}}_{N-1}^k \right\}$$

that solves the problem

$$\begin{aligned} \min_{\boldsymbol{\pi}^k} \quad & \mathbb{E}\left[M(\mathbf{s}_N) + \sum_{l=k}^{N-1} L(\mathbf{s}_l, \boldsymbol{\pi}_l^k(\mathbf{s}_l)) \right] \\ \text{s.t.} \quad & \mathbb{P}[\mathbf{s}_{k+1,\ldots,N} \in \mathbb{S} \,|\, \mathbf{s}_k, \boldsymbol{\pi}^k] \geq S_k \end{aligned}$$

(π^k are parameterized in practice such that it lies in the subspace of its actual admissible set)

- The control inputs applied to the closed-loop system are given by

$$\mathbf{u}_k = \tilde{\boldsymbol{\pi}}_k^k(\mathbf{s}_k), \;\; k = 0, \ldots, N-1$$

How to ensure recursive feasibility?

How to fulfill the MWPS constraint of the closed-loop system?

We will answer these questions by designing the parameter S_k properly

Recursive Feasibility of MWPS Constraints

Proposition 1: Assume that there exist feasible $\boldsymbol{\pi}^0$ satisfies the MWPS constraint:

$$\mathbb{P}[\mathbf{s}_{1,\dots,N} \in \mathbb{S} \,|\, \mathbf{s}_0, \boldsymbol{\pi}^0] \geq S_0 \geq S$$

and that each policy sequence $\boldsymbol{\pi}^k$ is built under the constraint:

$$\mathbb{P}[\mathbf{s}_{k+1,\dots,N} \in \mathbb{S} \,|\, \mathbf{s}_k, \boldsymbol{\pi}^k] \geq S_k \tag{1}$$

with $S_k = \gamma_k \mathbb{P}[\mathbf{s}_{k+1,\dots,N} \in \mathbb{S} \,|\, \mathbf{s}_k, \boldsymbol{\pi}^{k-1}]$, where $\gamma_k \in (0, 1]$, for all k. Then the closed-loop MWPS constraints under $\{\tilde{\boldsymbol{\pi}}_0^0, \dots, \tilde{\boldsymbol{\pi}}_{N-1}^{N-1}\}$ reads as:

$$\mathbb{P}\left[\mathbf{s}_{1,\dots,N} \in \mathbb{S} \,\middle|\, \mathbf{s}_0, \{\tilde{\boldsymbol{\pi}}_0^0, \dots, \tilde{\boldsymbol{\pi}}_{N-1}^{N-1}\}\right] \geq \prod_{k=1}^{N-1} \gamma_k S_0 \,.$$

Corollary.

- the choice of $\prod_{k=1}^{N-1} \gamma_k S_0 = S$ fulfills the closed-loop MWPS constraint.
- constraint (1) is feasible for $\boldsymbol{\pi}^k = \boldsymbol{\pi}^{k-1}$, i.e., the SMPC controller is recursive feasible.

Stochastic Linear MPC with MWPS guarantee

The linear system: $\mathbf{s}_{k+1} = A\mathbf{s}_k + B\mathbf{u}_k + \mathbf{w}_k$

The safe set: $\mathbb{S}$ is polytopic, i.e., $\mathbb{S} = \{\, \mathbf{s} \,|\, C\mathbf{s} + \mathbf{c} \leq 0\}$

At current state $\mathbf{s}_k$, for all $t = k, \ldots, N-1$, solve the problem

$$\begin{aligned}
\min_{\bar{\mathbf{u}}_{k,\ldots,N-1}} \quad & \mathbb{E}\left[\mathbf{s}_N^\top Q_N \mathbf{s}_N + \sum_{t=k}^{N-1} \left(\mathbf{s}_t^\top Q \mathbf{s}_t + \mathbf{u}_t^\top R \mathbf{u}_t\right)\right] \\
\text{s.t.} \quad & \bar{\mathbf{s}}_k = \mathbf{s}_k \\
& \bar{\mathbf{s}}_{t+1} = A\bar{\mathbf{s}}_t + B\bar{\mathbf{u}}_t, \qquad \text{(nominal dynamic model)} \\
& \mathbf{e}_{t+1} = (A + BK)\,\mathbf{e}_t + \mathbf{w}_t, \qquad \text{(error dynamic model)} \\
& \mathbf{s}_{t+1} = \bar{\mathbf{s}}_{t+1} + \mathbf{e}_{t+1}, \\
& \mathbb{P}[C\mathbf{s}_{t+1} + \mathbf{c} \leq 0,\ \forall t] \geq S_k \,,
\end{aligned}$$

where Q, Q_N are semi-positive definite, R is positive definite.

The policy $\boldsymbol{\pi}^k$ is parameterized as $\boldsymbol{\pi}_t^k(\mathbf{s}_t) := \bar{\mathbf{u}}_t + K\mathbf{e}_t$ (classic in stochastic/robust MPC)

Stochastic Linear MPC with MWPS guarantee

Cost function. assume zero-mean of $\mathbf{w}_k$, cost function reduces to

$$\bar{\mathbf{s}}_N^\top Q_N \bar{\mathbf{s}}_N + \sum_{t=k}^{N-1} \left(\bar{\mathbf{s}}_t^\top Q \bar{\mathbf{s}}_t + \bar{\mathbf{u}}_t^\top R \bar{\mathbf{u}}_t \right) + \sigma,$$

where σ is a constant term.

Chance constraint. reformulated as

$$\mathbb{P}[H(\mathbf{w}_{k,\ldots,N-1}) + \mathcal{C}\bar{\mathbf{s}}_{k+1,\ldots,N}^\top \leq \mathbf{0}] \geq S_k$$

with appropriate matrices H and $\mathcal{C}$.

Scenario Approximation. further reformulated via samples

$$H^{(i)}(\mathbf{w}_{k,\ldots,N-1}^{(i)}) + \mathcal{C}\bar{\mathbf{s}}_{k+1,\ldots,N}^\top \leq \mathbf{0}$$

(for all $i = 1, \ldots, N_k$, and N_k is the number of samples [cf. Calafiore 2009])

Pick up a "representative" sample. label:

$$\mathcal{I}_j = \max_{i \in \mathbb{I}_{[1,N_k]}} [H^{(i)}]_j, \quad \forall j \in \mathbb{I}_{[1,n_c]},$$

(n_c is the number of constraints generated by a sample i)

Stochastic Linear MPC with MWPS guarantee

Eventually, we derive the following QP

$$\min_{\bar{\mathbf{u}}} \bar{\mathbf{s}}_N^{\mathrm{T}} Q_N \bar{\mathbf{s}}_N + \sum_{t=k}^{N-1} \left(\bar{\mathbf{s}}_t^{\mathrm{T}} Q \bar{\mathbf{s}}_t + \bar{\mathbf{u}}_t^{\mathrm{T}} R \bar{\mathbf{u}}_t \right)$$

$$\text{s.t. } \bar{\mathbf{s}}_k = \mathbf{s}_k$$

$$\bar{\mathbf{s}}_{t+1} = A\bar{\mathbf{s}}_t + B\bar{\mathbf{u}}_t + \bar{\mathbf{w}}_t,$$

$$\mathcal{I}_j + [\mathcal{C}]_j \bar{\mathbf{s}}_{k+1,\ldots,N} \leq 0$$

A regular QP of the same complexity as a normal linear MPC

Algorithm: linear SMPC with MWPS constraint

Initialization: S_0, $\gamma_{1,\ldots,N-1}$, initial state $\mathbf{s}_0$;
while $k = 0 : N-1$ **do**
  Evaluate S_k through Monte Carlo simulation;
  Generate N_k samples;
  Get the solution $\bar{\mathbf{u}}^*_{k,\ldots,N-1}$ by solving the QP above;
  Send $\bar{\mathbf{u}}^*_k$ to the actual system and update state: $\mathbf{s}_{k+1} = A\mathbf{s}_k + B\bar{\mathbf{u}}^*_k + \mathbf{w}_k$;
end

Stochastic Linear MPC with MWPS guarantee: a case study

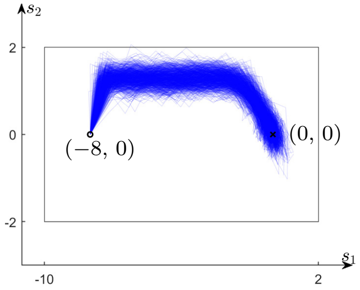

Figure: State trajectories via running Monte Carlo simulations

- System matrices:
  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$
- Disturbance: $\mathbf{w}_k \sim \mathcal{N}(0, 0.04 \cdot I)$
- Safe set matrices:
  $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} -2 \\ -2 \\ -10 \\ -2 \end{bmatrix}$
- $Q = I,\ R = 0.1,$
  $K = [-0.6167, -1.2703]$
  $Q_N = \begin{bmatrix} 2.0599 & 0.5916 \\ 0.5916 & 1.4228 \end{bmatrix}$

- Set $N = 11$, $S_0 = 0.98$ and $\gamma_{1,\ldots,10} = 0.99$, resulting in $S = \prod_{k=1}^{10} \gamma_k S_0 = 0.8863$
- Running 10^5 missions shows that the resulting ratio of mission success is 99.88%

This discrepancy is due to that the scenario-based method adopted is conservative.

Summary

Conclusions

- Showed that the remaining MWPSs remain constant in the expected value sense
- Proposed a recursively feasible control scheme while ensuring MWPS constraint
- Deployed the idea in the linear case via an efficient scenario-based approach

Future work:

- More advanced methods in place of the classic Monte-Carlo sampling
- Receding-horizon policy in place of the shrinking-horizon policy

Contact: `kai.wang@ntnu.no`

Thank you! Questions?